Math 411, Fall 1997

Solutions to the Second Midterm

Problem 1. Is there a two point compactification of an open disk? More precisely, does there exist a compact Hausdorff space X such that:

(1) X contains a dense subspace A homeomorphic to an open disk;
(2) $X \setminus A$ consists of two points?

(*Hint*: X should be Hausdorff. If you cannot give a rigorous proof, which is a bit hard, though welcome, try to explain your opinion.)

Solution: No, there isn't. Suppose X is such a space, and ∞_1, ∞_2 are the two points of $X \setminus A$. We will prove that these two points cannot be separated (and, hence, X is not Hausdorff). Let U_1, U_2 be some disjoint neighborhood of ∞_1, ∞_2, respectively. Eventually, we are going to prove that this assumption contradicts to the connectedness of the "boundary" of A. Put $F_i = A \setminus U_i$. Then $F_1 \cap F_2 = (X \setminus U_1) \cap (X \setminus U_2)$ is closed in X, hence compact, and $F_1 \cup F_2 = A$ (since U_1, U_2 are disjoint). Besides, neither of the F_i's is compact. (Indeed, suppose that, say, F_1 is compact, hence closed in X. Then, since A is dense, we have: $\operatorname{Cl} A = \operatorname{Cl}(F_1 \cup F_2) = F_1 \cup \operatorname{Cl} F_2 = X$, i.e., $\infty_1, \infty_2 \in \operatorname{Cl} F_2$ and $U_2 \subset X \setminus \operatorname{Cl} F_2$ does not contain ∞_2.) Thus, we represented A, which we now regard as the standard open unit disk in $\mathbb{R}^2$, as the union of two closed noncompact subsets with compact intersection. Let $S^1 = \partial A$ (in $\mathbb{R}^2$). Since $F_1 \cap F_2$ is compact (and, in particular, closed in $\mathbb{R}^2$), $\operatorname{dist}(F_1 \cap F_2, S^1) = \varepsilon > 0$. Let B be the annulus $\{\mathbf{x} \in \mathbb{R}^2 \mid 1 - \varepsilon/2 < |\mathbf{x}| < 1\}$. It is connected, but, on the other hand, it is disjoint union of two non-empty closed sets $F_1 \cap B$ and $F_2 \cap B$. (The sets are disjoint because, by our construction, $F_1 \cap F_2 \cap B = \varnothing$, and they are non-empty since any closed set that lies entirely in $A \setminus B$ should be compact.) This is a contradiction.

Problem 2. Let X be a topological space. Two points $x, y \in X$ are said *to lie in the same quasicomponent of* X if there is no separation $X = A \cup B$ of X into two disjoint nonempty open sets with $x \in A$ and $y \in B$.

(1) Show that 'to lie in the same quasicomponent' is an equivalence relation and, thus, X splits into disjoint union of quasicomponents.
(2) Show that each component of X belongs to a quasicomponent.
(3) Show that the quasicomponent of a point $x \in X$ is the intersection of all open-closed subsets containing x.
(4) Does a quasicomponent need to be closed? Open? Why?

Solution: (1) Reflexivity and symmetricity are obvious. For the transitivity assume that $x \sim y$, $y \sim z$, and there is a separation $X = A \cup B$ with $x \in A$ and $z \in B$. Then either $y \in A$ and, hence, $x \not\sim y$, or $y \in B$ and, hence, $y \not\sim z$.

(2) Let C be a component, $x, y \in C$, and $x \not\sim y$. Then there is a separation $X = A \cup B$ with $x \in A$ and $y \in B$, and $C \cap A$ and $C \cap B$ form a separation of C, which, thus, is not connected.

(3) Let Q be a quasicomponent, $x \in Q$, and A_α all open-closed sets containing x. If $Q \not\subset A_\alpha$ for some α, then $X = A_\alpha \cup (X \setminus A_\alpha)$ is a separation of X such that $Q \cap (X \setminus A_\alpha)$ is not empty, and for any point $y \in Q \cap (X \setminus A_\alpha)$ one has $x \not\sim y$. Conversely, let $y \in \bigcap_\alpha A_\alpha$ and $y \not\sim x$. Then there is a separation $X = A \cup B$ with $x \in A$ and $y \notin A$. But A is open-closed, i.e., one of A_α.

(4) Due to (3) a quasicomponent is closed as intersection of closed sets. It does not need to be open: say, the quasicomponents of $\mathbb{Q}$ are single points. (Why?)

Problem 3. True or false: the intersection $A = \bigcap_{i=1}^{\infty} A_i$ of a decreasing sequence $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of connected subsets of a topological space is connected?

Solution: The statement is **false**. Let $A_n = \mathbb{R}^2 \setminus B_n$, where $B_n = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1,\ y > n\}$. Then all A_n are connected, $A_n \supset A_{n+1}$, and $\bigcap_n A_n = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 1 \text{ or } x \leqslant -1\}$, which is disconnected.