Math 411, Spring 1997

Solutions to the First Midterm

Problem 1. (1) Let $\mathbb{Q} \subset \mathbb{R}$ be the set of rationals ($\mathbb{R}$ being considered in its standard metric topology). What are $\operatorname{Cl}\mathbb{Q}$ and $\operatorname{Int}\mathbb{Q}$?

(2) The same question for the subspace $A \subset \mathbb{R}^2$ obtained from the standard open unit disk by deleting the radius $\{0 \leqslant y < 1\}$.

Solution: (1) $\operatorname{Cl}\mathbb{Q} = \mathbb{R}$, since every real has a rational approximation (i.e., can be represented as the limit of a sequence of rationals). $\operatorname{Int}\mathbb{Q} = \varnothing$, since every interval in $\mathbb{R}$ has irrational points and, hence, does not entirely belong to $\mathbb{Q}$.

(2) A is open as difference between an open and a closed set. (Why?) Hence, $\operatorname{Int} A = A$. As to the closure, it is the closed disk D^2. (One can easily see that every point of this disk is a limit point of A.)

Problem 2. Let $B \subset X$ be a subset of a topological space X. Prove that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} B = \operatorname{Cl}\operatorname{Int} B$ and $\operatorname{Int}\operatorname{Cl}\operatorname{Int}\operatorname{Cl} B = \operatorname{Int}\operatorname{Cl} B$. Can one strip anything off from these identities? (E.g., is it always true that $\operatorname{Int}\operatorname{Cl}\operatorname{Int} B = \operatorname{Int} B$, or $\operatorname{Int}\operatorname{Cl} B = B$?) As usual, you are supposed to either prove an assertion, or give a counterexample. (*Hint*: The first question is easy if you choose an appropriate definitions of Cl and Int. As to the second one, you can use, e.g., Problem 1.)

Solution: The above appropriate definitions are "the smallest closed set ... " and "the largest open set ... ". In particular, from this it obviously follows that $A \subset B$ always implies $\operatorname{Cl} A \subset \operatorname{Cl} B$ and $\operatorname{Int} A \subset \operatorname{Int} B$. Besides, $\operatorname{Cl}\operatorname{Cl} A = \operatorname{Cl} A$ and $\operatorname{Int}\operatorname{Int} A = \operatorname{Int} A$ for any subset $A \subset X$. Now, let us prove, say, that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A = \operatorname{Cl}\operatorname{Int} A$. We have:

$$\begin{aligned}
\operatorname{Int}(\operatorname{Cl}\operatorname{Int} B) \subset \operatorname{Cl}\operatorname{Int} B &\implies \operatorname{Cl}(\operatorname{Int}\operatorname{Cl}\operatorname{Int} B) \subset \operatorname{Cl}(\operatorname{Cl}\operatorname{Int} B) = \operatorname{Cl}\operatorname{Int} B,\\
\operatorname{Cl}(\operatorname{Int} B) \supset \operatorname{Int} B &\implies \operatorname{Cl}\operatorname{Int}(\operatorname{Cl}\operatorname{Int} B) \supset \operatorname{Cl}\operatorname{Int}(\operatorname{Int} B) = \operatorname{Cl}\operatorname{Int} B,
\end{aligned}$$

and the two inclusions imply the desired identity.

To construct a counterexample to the other conjectures, let us take for X the disjoint union $\mathbb{R}^2 \cup \mathbb{R}^1 \cup \mathbb{R}^1$, and for $B \subset X$, the union of the set $A \subset \mathbb{R}^2$ (see Problem 1(2)), $\mathbb{Q} \subset \mathbb{R}^1$, and $\mathrm{pt} \subset \mathbb{R}^1$ (a single point). Then we have:

$$\begin{aligned}
B &= A \qquad \cup\ \mathbb{Q} \quad \cup\ \mathrm{pt},\\
\operatorname{Cl} B &= D^2 \qquad \cup\ \mathbb{R}^1 \ \cup\ \mathrm{pt},\\
\operatorname{Int}\operatorname{Cl} B &= \{\text{open disk}\} \ \cup\ \mathbb{R}^1,\\
\operatorname{Cl}\operatorname{Int}\operatorname{Cl} B &= D^2 \qquad \cup\ \mathbb{R}^1,\\
\operatorname{Int} B &= A,\\
\operatorname{Cl}\operatorname{Int} B &= D^2,\\
\operatorname{Int}\operatorname{Cl}\operatorname{Int} B &= \{\text{open disk}\}\,,
\end{aligned}$$

and one can see that all the seven sets are different.

Problem 3. Prove that the circle $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ is connected. Prove that S^1 is not homeomorphic to the unit interval $I = [0,1]$. (*Hint*: Represent S^1 as a continuous image of a space known to be connected. For the second question, try to remove a point and compare the results.)

SOLUTION: The map $I \to S^1$, $t \mapsto \exp(2\pi i t)$ is continuous and onto. I is connected, hence so is S^1. After deleting a point S^1 is still connected (as the complement is homeomorphic to $(0,1)$). On the other hand, in I there are points (any interior point) deleting which makes the space disconnected. Hence, $S^1 \not\cong I$.

Problem 4. Give a topological classification of the latin letters, assuming that they appear like this:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

(I.e., you are supposed to (a) split the alphabet into classes of homeomorphic letters, **and** (b) prove that letters of distinct classes are not homeomorphic.)

SOLUTION: The homeomorphism classes of the letters, along with the invariants which distinguish between them, are given in the following table: (See also comments below)

N	Class	Endpoints	$I_1(X)$	$I_2(X)$	$\#I_3(X)$	$\#I_4(X)$
1	A R	2	$\mathrm{pt} \cup \mathrm{pt} \cup I^\circ \cup I^\circ$	$\hat{I} \cup \hat{I}$	0	0
2	B	0	X itself	$\varnothing$	0	0
3	C G I J L M N S U V W Z	2	$\mathrm{pt} \cup \mathrm{pt}$	I°	0	0
4	D O	0	X itself	$\varnothing$	0	0
5	E F T Y	3	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ$	1	0
6	H K	4	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ \cup I^\circ \cup I^\circ$	2	0
7	P	1	$\mathrm{pt} \cup I^\circ$	$\hat{I}$	0	0
8	Q	2	$\mathrm{pt} \cup \mathrm{pt} \cup I^\circ$	$I^\circ \cup I^\circ$	1	0
9	X	4	$\mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt} \cup \mathrm{pt}$	$I^\circ \cup I^\circ \cup I^\circ \cup I^\circ$	0	1

Here pt, I°, and $\hat{I}$ stand, respectively, for a single point, open interval $(0,1)$, and semi-open interval $(0,1]$, and $\cup$ denotes disjoint union of topological spaces. In order to distinguish the classes, introduce the subsets $I_n(X)$ of the points $x \in X$ such that $X \smallsetminus \{x\}$ consists of n connected components. Also, let us call a point $x \in X$ an *endpoint* of X if $U \smallsetminus \{x\}$ is connected for any connected neighborhood U of x. Clearly, any homeomorphism $X \to Y$ must take $I_n(X)$ to $I_n(Y)$ and the endpoints of X to those of Y.

In view of the table the only problem left is to prove that classes 2 and 4 are different. For this purpose just note that any *pair* of distinct points divides letters D and O (which are obviously homeomorphic to S^1) into two intervals, while in B there is a pair of points which divides it into three intervals.

Remark. Note that the comma "," belongs to class 3, since it is also a segment (though a very small one). The other punctuation marks form three more classes: {.}, {:}, and {;, !, ?}.

Remark. Note that this classification depends on the font one uses. Try, for example, these letters: (Computer modern typewriter font by D. Knuth. Just for the record: what we did was Computer modern sans serif)

`A B C D E F G H I J K L M N O P Q R S T U V W X Y Z`