HOMEWORKS

Homework 7 (due on 17/05)

Problem 1. Let $p\colon X \to B$ be a covering, and let $G = p_*\pi_1(X) \subset \pi_1(B)$ be its group. An *automorphism* of p (or a *deck translation*) is a continuous self map $f\colon X \to X$ such that $p \circ f = p$. Prove that:

(1) the set $\operatorname{Aut} p$ of automorphisms of p is a group; an element of $\operatorname{Aut} p$ is uniquely determined by the image of one point;
(2) $\operatorname{Aut} p$ is canonically isomorphic to $N(G)/G$, where $N(G)$ is the normalizer of G in $\pi_1(B)$;
(3) $\operatorname{Aut} p$ acts transitively on the fiber $p^{-1}(b_0)$, $b_0 \in B$, if and only if $G \subset \pi_1(B)$ is normal (topologists call such coverings *regular*; algebraic geometers call them *Galois*).

Homework 6 (due on 20/04)

Problem 1. Read the proof of the 5 color theorem in the textbook (§4.2). How many gaps can you find? Fill them all. Try to use the same approach to prove the 4 color theorem. Where does it fail?

Problem 2. Analyze the proof and explain why it does not work for $\mathbb{R}\mathrm{p}^2$. Show that the chromatic number of $\mathbb{R}\mathrm{p}^2$ equals 6 and that of torus equals 7. (*Hint*: use Homework 5.)

Homework 5 (due on 13/04)

Problem 1. Does there exist a triangulation of a torus with exactly 14 triangles? Does there exist a triangulation of $\mathbb{R}\mathrm{p}^2$ with exactly 10 triangles?

Problem 2. Investigate regular graphs on an arbitrary closed surface. (Try to say as much as you can.)

Homework 4 (due on 23/03)

Problem 1. Let Δ^n be the standard n-simplex. Use the standard triangulation of Δ^n to prove that $H_0(\Delta^n) = \mathbb{Z}$ and $H_i(\Delta^n) = 0$ for $i > 0$. *Hint*: if $\mathbb{Z}$-coefficients are too difficult, try to do that over a field. The magic words (from linear algebra) to dig internet for are *exterior power*.

Homework 3 (due on 2/03)

Problem 1. Show that the tangent to a projective curve C given by a homogeneous equation $f(x, y, z) = 0$ at a point $[x_0 : y_0 : z_0] \in C$ is given by $xf_x + yf_y + zf_z = 0$, where all partials are evaluated at (x_0, y_0, z_0). *Hint*: pass to an affine part and use calculus. The following *Euler's formula* may help: if $f(x, y, z)$ is a homogeneous polynomial of degree d, then $xf_x + yf_y + zf_z \equiv d \cdot f$. Prove this as well.

Problem 2. Let C be a projective conic (a curve of degree two) given by the equation $\mathbf{x}^{\mathrm{T}} M \mathbf{x} = 0$, $\mathbf{x} = [x : y : z]$, for some symmetric (3×3)-matrix M. Assume that C is nonsingular, *i.e.*, $\det M \neq 0$.

(1) Prove that the set of all tangents to C forms a certain conic $C^\vee$ in the dual projective plane. It is called the *dual* of C.

(2) Prove that the original conic C is the dual of $C^\vee$.
(3) State accurately the duality between points in and lines tangent to the curves C and $C^\vee$.

Hint: first, use Problem 1 to show that the line tangent to C at a point $\mathbf{x}_0 \in C$ is given by $\mathbf{x}^{\mathrm{T}} M \mathbf{x}_0 = 0$.

Homework 2 (due on 23/02)

Problem 1. Give a topological (*i.e.*, up to homeomorphism) classification of the capital Latin letters

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

(here, in the Computer Modern Sans Serif font; try other fonts as well).

Homework 1 (due on 16/02)

Problem 1. Let X, Y be topological spaces and let $f_i \colon A_i \to Y$, $i = 1, 2$, be continuous maps defined on two subsets $A_1, A_2 \subset X$. Assume that the restrictions of f_1 and f_2 to the intersection $A_1 \cap A_2$ coincide, and define the map $f \colon A_1 \cup A_2 \to Y$ via $f(x) = f_1(x)$ if $x \in A_1$ and $f(x) = f_2(x)$ if $x \in A_2$.

(1) Prove that if A_1 and A_2 are *both open* or *both closed*, then f is continuous.
(2) Illustrate that the condition above is important.
(3) Try to generalize the statement to infinitely many subsets (all closed or all open). Be careful: an extra condition may be required!

Problem 2. Let X be a topological space and $A_\alpha \subset X$ a family of connected subspaces. Prove that, if $\bigcap_\alpha A_\alpha \neq \varnothing$, then $\bigcup_\alpha A_\alpha$ is connected.

Problem 3. Let X be a topological space. Define the (*connected*) *component* of a point $x \in X$ as the union of all connected subsets containing X.

(1) Prove that connected components are connected.
(2) Prove that any two components either coincide or are disjoint, so that X is a disjoint union of connected components. (In other words, the relation *to be in the same component* is an equivalence relation.)
(3) Prove that components are closed. Do they need to be open?
(4) Prove that the closure of a connected subset $A \subset X$ is connected.