Math 310 Topology, Spring 2010

Solutions to Midterm 2

Problem 1. Let $\mathrm{Ch}_G(\lambda)$ be the chromatic polynomial of a graph G. Prove that:

(1) $\deg \mathrm{Ch}_G$ is the number of vertices of G;
(2) $\mathrm{Ch}_G(0) = 0$;
(3) Ch_G has no negative real roots.

Solution: All three statements follow easily by induction in the number of edges (starting from $\mathrm{Ch}_{V_n}(\lambda) = \lambda^n$, where V_n is the graph with n vertices and no edges), using the recursive formula $\mathrm{Ch}_G = \mathrm{Ch}_{G-e} - \mathrm{Ch}_{G/e}$, where e is any edge of G and G/e is obtained from G by contracting e (and identifying appropriate vertices and edges). For item (3), note that the absence of negative real roots of a polynomial P means that, for any real $\lambda < 0$, the sign of $P(\lambda)$ is $(-1)^{\deg P}$, and when two such polynomials of degrees n and $n-1$ are **subtracted**, the property is preserved.

For items (1) and (2), one can also argue that, by definition, for any graph G with n vertices and any integer $\lambda \geqslant 0$ one has $\lambda^n = \mathrm{Ch}_{V_n}(\lambda) \geqslant \mathrm{Ch}_G(\lambda) \geqslant \mathrm{Ch}_{K_n}(\lambda) = \lambda(\lambda-1)\dots(\lambda-n+1)$.

Note that most other speculations are not true. For example, there are chromatic polynomials that have some roots non-real.

Problem 2. Let G be a loop free graph with n vertices and without multiple edges, and assume that $\deg x + \deg y \geqslant n-1$ for any pair of distinct vertices x, y. Prove that G has a *Hamiltonian cycle*, *i.e.*, a cycle that visits each vertex exactly once. Here is a possible way of attacking this problem (where the words 'path' and 'cycle' assume that each vertex is visited *at most* once):

(1) Prove that G is connected.
(2) Consider a *maximal* path (*i.e.*, one that cannot be enlarged from either end) and show that it can be converted to a cycle on the same subset of vertices.
(3) Show that any *cycle* as above can be 'opened' and enlarged to a path containing more vertices.

Solution: (1) By the pigeonhole principle, any two distinct vertices, if not connected by an edge, have a common neighbor, hence are connected by a path of length two.

(2) Let $\gamma\colon v_1 \to v_2 \to \dots \to v_r$ be a maximal path. (We order the vertices along γ.) In other words, we assume that neither v_1 nor v_r is connected to any other vertex that is not in γ, as otherwise we would be able to extend γ to a longer path. Then the pigeonhole principle implies (almost directly) that there is a pair v_k, v_{k+1} of **consecutive** vertices of γ such that v_1 is connected to v_{k+1} and v_k is connected to v_r. Then

$$\delta\colon v_1 \to v_{k+1} \to (\text{forwards along } \gamma) \to v_r \to v_k \to (\text{backwards along } \gamma) \to v_1$$

is a cycle on the same set of vertices $v_1 \dots, v_r$. For the next step of the proof, we reorder the vertices along δ. (Note that the new order may differ from the previous one!)

(3) If the cycle δ constructed above is not Hamiltonian, pick a vertex u that is not in δ. By (1), there is a path connecting u to a vertex in δ. Let $u = u_1 \to \dots \to u_m = v_s \in \delta$ be such a path. Then

$$u_1 \to \dots \to u_m = v_s \to (\text{forwards along } \delta) \to v_r \to v_1 \to (\text{forwards along } \delta) \to v_{s-1}$$

is a path longer than γ, and one can continue steps (2) and (3).

Problem 3. Let S be a compact surface with one boundary component.

(1) Compute the homology $H_*(S)$ and the homomorphisms $H_*(\partial S) \to H_*(S)$ induced by the inclusion $\partial S \hookrightarrow S$.
(2) Generalize the Borsuk theorem: S does not retract to ∂S.

(Recall that a *retraction* is a map $\rho\colon S \to \partial S$ whose restriction to ∂S is the identity. The original Borsuk theorem deals with the case when S is a disk.)

Remark. In fact, this statement holds for *any* compact manifold (of any dimension with any boundary): a compact manifold does not retract to its boundary.

Solution: (1) Consider the standard plane model (a polygon with pairs of edges identified) and make a hole at one of the corners. The result is a polygon P with all but one edges split into identified pairs and one edge, call it c, free. As usual, after the identifications, there is a single vertex left; call it v. Then the complex is

$$0 \to \mathbb{Z}[P] \xrightarrow{\partial_2} \mathbb{Z}c \oplus \bigoplus_{i=1}^{g}(\mathbb{Z}a_i \oplus \mathbb{Z}b_i) \xrightarrow{\partial_1} \mathbb{Z}v \to 0 \qquad \text{or} \qquad 0 \to \mathbb{Z}[P] \xrightarrow{\partial_2} \mathbb{Z}c \oplus \bigoplus_{i=1}^{q}\mathbb{Z}a_i \xrightarrow{\partial_1} \mathbb{Z}v \to 0$$

(for orientable and non-orientable surfaces, respectively). In both cases, $\partial_1 = 0$, and the map ∂_2 is found as the boundary of P: in the orientable case, $\partial_2[P] = c$, in the non-orientable case, $\partial_2[P] = 2\sum_i a_i + c$. Thus, one has $H_0(S) = \mathbb{Z}$, $H_2(S) = 0$ (as $\partial_2[P] \neq 0$), and $H_1(S) = \bigoplus_{i=1}^{g}(\mathbb{Z}a_i \oplus \mathbb{Z}b_i)$ or $H_1(S) = \bigoplus_{i=1}^{q}\mathbb{Z}a_i$. On the other hand, $H_0(\partial S) = \mathbb{Z}$ and $H_1(\partial S) = \mathbb{Z}c$. Clearly, the inclusion homomorphism is an isomorphism in dimension 0, and it sends c to c in dimension 1; in view of the relations above (whatever is in the image of ∂_2 should be set equal to zero), one has $\mathrm{in}_* c = 0$ in the orientable case and $\mathrm{in}_* c = -2\sum a_i$ in the nonorientable case.

For item (2), it is easier to consider the homology with coefficients $\mathbb{Z}_2$: then, in all cases one has $\mathrm{in}_* c = 0$, *i.e.*, $\mathrm{in}_* = 0$. An informal, but geometrically transparent explanation is as follows: $\mathrm{in}_* c = 0$ because c bounds the surface itself.

(2) This is, indeed, a direct generalization. Assume that there is a retraction $\rho\colon S \to \partial S$. Then $\rho \circ \mathrm{in} = \mathrm{id}$ and, passing to the homology with coefficients $\mathbb{Z}_2$, one obtains $\rho_* \circ \mathrm{in}_* = \mathrm{id}_{\mathbb{Z}_2}$. This contradicts to item (1), where it is shown that $\mathrm{in}_* = 0$. (Note that, in general, $H_1(S;\mathbb{Z}_2) \neq 0$!)

Problem 4. Prove that:

(1) A closed surface is not homotopy equivalent to a *CW*-complex of dimension $\leqslant 1$.
(2) A compact surface with **nonempty boundary** is homotopy equivalent to a graph. (*Hint*: Try to find a more or less explicit deformation retraction.)

SOLUTION: (1) For a closed connected surface S one has $H_2(S;\mathbb{Z}_2) = \mathbb{Z}_2$ (computed in class), whereas for any *CW*-complex X of dimension d one has $H_i(X;G) = 0$ for any $i > d$ and any coefficient group G (an immediate corollary of the cellular definition of homology). Due to the homotopy invariance of the homology, a *CW*-complex of dimension $\leqslant 1$ (or any space homotopy equivalent to it) cannot have $H_2 \neq 0$.

(2) Consider the standard plane model of a surface (a polygon P with pairs of edges identified) and make one hole at the center of P. The result is a disk with a hole, which retracts to its boundary (basically, the map $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$). This is a deformation retraction, which retracts a surface S with one boundary component to a graph in S (the image of ∂P). If there is more than one hole, connect all but one of them to ∂P by extra edges and cut along these edges. The result is again a polygon with some pairs of edges identified and a hole in the middle, and the same construction as above works.