Math 240-01, Spring 2000

Solutions to the Final Exam

Problem 1. A 1-kg weight is attached to the lower end of a spring suspended from the ceiling. In its equilibrium position the weight stretches the spring 10 cm. Starting at time $t_0 = 0$ the weight is pulled down by a force of 30 N for π sec and then released. Find the displacement of the weight as function of time and the amplitude of the resulting free oscillation. (Assume $g = 10$.)

SOLUTION: Introduce the constants: $m = 1$ kg is the mass of the weight, $y_0 = 0.1$ m, the original displacement in the equilibrium position, k, the spring coefficient, $F_0 = 30$ N, the value of the force applied to the weight, and $a = \pi$ sec, the duration of the force. From Hook's law we have $mg = ky_0$, i.e., $k = mg/y_0$. The force applied as function of time can be given by $F(t) = F_0(1 - \alpha(t - a))$, where α is the step function. Newton's second law gives the equation $my'' = -k(y + y_0) + mg + F(t)$, $y(0) = 0$, $y'(0) = 0$, or $y'' + (k/m)y = (F_0/m)(1 - \alpha(t - a))$. (Here y is the displacement; the axis is directed straight down and the origin is chosen at the equilibrium position.) Let $\hat{y} = L\{y\}$. Apply the Laplace transform:

$$s^2\hat{y} + \frac{k}{m}\hat{y} = \frac{F_0}{ms}(1 - e^{-as}), \quad \text{or} \quad \hat{y} = \frac{F_0/m}{s(s^2 + (k/m))}(1 - e^{-as}).$$

Hence (using properties of L^{-1}), $y = G(t) - G(t - a)\alpha(t - a)$, where $G(t)$ is the inverse transform of $g(s) = F_0/[ms(s^2 + (k/m))]$. To find $G(t)$, use the tables:

$$g(s) = \frac{F_0}{k}\left[\frac{1}{s} - \frac{s}{s^2 + (k/m)}\right], \quad \text{whence} \quad G(s) = \frac{F_0}{k}\left(1 - \cos\sqrt{\frac{k}{m}}t\right).$$

Finally, substituting the constants, we get

$$\boxed{y(t) = 0.3\big(1 - \alpha(t - \pi)\big) - 0.3\big(\cos 10t - \cos 10(t - \pi)\alpha(t - \pi)\big)}.$$

When the force is removed (i.e., $t > \pi$), one has $\alpha(t - \pi) = 1$ and, since $\cos 10(t - \pi) = \cos 10t$, we get $y(t) = 0$ for $t > \pi$, i.e., the weight comes back to the equilibrium position and there is no oscillation! (Thus, the amplitude is $\boxed{0}$.) Surprising, isn't it?

Problem 2. Represent the general solution to $(x^2 - 3)y'' + 6xy' + 6y = 0$ by a power series about the origin. Indicate the interval of convergence of the series.

SOLUTION: Let $y = \sum_{n=0}^{\infty} c_n x^n$ be the series in question. Plug in to get

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - 3\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + 6\sum_{n=0}^{\infty} nc_n x^n + 6\sum_{n=0}^{\infty} c_n x^n = 0,$$

or, after shifting the index in the second sum, $\sum_{n=0}^{\infty}[n(n-1)c_n - 3(n+2)(n+1)c_{n+2} + 6nc_n + 6c_n]x^n = 0$. Equating the coefficients to 0, we get the recurrence relation $(n^2 + 5n + 6)c_n = 3(n+2)(n+1)c_{n+2}$, or $c_{n+2} = (n+3)c_n/3(n+1)$. Shift the index back: $c_n = (n+1)c_{n-2}/3(n-1)$, $n \geqslant 2$. This obviously yields

$$c_{2k} = \frac{(2k+1)(2k-1)\dots(3)}{3^k(2k-1)(2k-3)\dots(1)}c_0 = \frac{2k+1}{3^k}c_0 \quad \text{and} \quad c_{2k+1} = \frac{(2k+2)(2k)\dots(4)}{3^k(2k)(2k-2)\dots(2)}c_1 = \frac{k+1}{3^k}c_1.$$

Thus, the solution is

$$\boxed{y = c_0\sum_{k=0}^{\infty}\frac{2k+1}{3^k}x^{2k} + c_1\sum_{k=0}^{\infty}\frac{k+1}{3^k}x^{2k+1}}.$$

The equation has two singular points, $x = \pm\sqrt{3}$. Hence, the interval of convergence of the series is $(-\sqrt{3}, \sqrt{3})$.

Problem 3. Find the inverse Laplace transforms of

$$\text{(a)}\quad f(s) = \frac{2s^2-2}{s^2(s^2-2s+2)}\,, \qquad \text{(b)}\quad f(s) = \frac{1}{s^2+6s+13}\,.$$

SOLUTION: (a) Since $s^2 - 2s + 2 = (s-1)^2 + 1$ has no real roots, the elementary fraction decomposition has the form

$$\frac{2s^2-2}{s^2(s^2-2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{(s^2-2s+2)}.$$

Get rid of the denominators, $2s^2 - 2 = As(s^2-2s+2) + B(s^2-2s+2) + (Cs+D)s^2$, and equate the coefficients: $A + C = 0$, $-2A + B + D = 2$, $2A - 2B = 0$, $2B = -2$. Hence, $A = B = -1$, $C = D = 1$, and

$$L^{-1}\left\{\frac{2s^2-2}{s^2(s^2-2s+2)}\right\} = -L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{(s-1)+2}{(s-1)^2+1}\right\} = \boxed{-1 - t + e^t\cos t + 2e^t \sin t}.$$

(b) This one is straightforward:

$$L^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = L^{-1}\left\{\frac{1}{(s+3)^2+4}\right\} = \boxed{\frac{1}{2}e^{-3t}\sin 2t}.$$

Problem 4. Find the first five terms of the power series expansion about $x_0 = 1$ of the solution to

$$y'' = (y')^3 + x^2, \qquad y(1) = 1, \quad y'(1) = 1.$$

SOLUTION: We need the values of y and its first four derivatives at 1. From the initial conditions we have $y(1) = 1$ and $y'(1) = 1$, and, differentiating the equation and plugging in, we obtain:

$$\begin{aligned} y'' &= (y')^3 + x^2, & y''(1) &= 2,\\ y''' &= 3(y')^2y'' + 2x, & y'''(1) &= 8,\\ y^{\text{IV}} &= 6y'(y'')^2 + 3(y')^2y''' + 2, & y^{\text{IV}}(1) &= 50. \end{aligned}$$

Thus, $y = \sum_{n=0}^{\infty} y^{(n)}(1)(x-1)^n/n! = \boxed{1 + (x-1) + (x-1)^2 + \frac{4}{3}(x-1)^3 + \frac{25}{12}(x-1)^4 + \dots}$.

Problem 5. Show that

$$L^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx.$$

SOLUTION: We have $L^{-1}\{1/\sqrt{s}\} = 1/\sqrt{\pi t}$ and $L^{-1}\{1/(s-1)\} = e^t$. Use the convolution theorem:

$$L^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} = \frac{1}{\sqrt{\pi}}\int_0^t e^{t-u}\frac{du}{\sqrt{u}} = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx \qquad \text{(substituting } u = x^2\text{)}.$$