Math 240-01, Spring 2000

Solutions to Midterm 2

Problem 1. Given that $y = x^2$ is a solution to $x^2y'' + 2xy' - 6y = 0$, find a general solution to the equation $x^2y'' + 2xy' - 6y = 1$.

SOLUTION: We will look for the solution in the form $y = x^2v$. Then $y' = 2xv + x^2v'$, $y'' = 2v + 4xv' + x^2v''$, and substitution gives $x^4v'' + 6x^3v' = 1$. This is a first order linear equation in $u = v'$; its solution is given by the formula: $u = 1/(3x^3) + C/x^6$. Hence, $v = \int u\,dx = -1/(6x^2) + C/x^5 + C_1$, and the answer is $y = x^2v = \boxed{-1/6 + C/x^3 + C_1x^2}$.

Problem 2. Let $y_1(x)$ and $y_2(x)$ be the solutions to the equation $x^2y'' + 2xy' + (1 - x^2)y = 0$ satisfying $y_1(1) = 1$, $y_1'(1) = 0$ and $y_2(1) = -3$, $y_2'(1) = 2$. Let, further, $W[y_1, y_2](x)$ be the Wronskian of y_1, y_2. Find $W[y_1, y_2](2)$.

SOLUTION: As known, the Wronskian is given by $W(x) = \exp\bigl(-\int p(x)\,dx\bigr) = \exp\bigl(-\int 2dx/x\bigr) = C/x^2$. The value at $x = 1$ and, hence, C are found from the initial conditions: $C/(1)^2 = \left|\begin{smallmatrix} 1 & -3 \\ 0 & 2 \end{smallmatrix}\right| = 2$. Thus, $C = 2$ and $W(2) = 2/(2)^2 = \boxed{1/2}$.

Problem 3. Find a general solution to $y'' - 3y' + 2y = e^x + \sin 2x$.

SOLUTION: First, solve the homogeneous equation $y'' - 3y' + 2y = 0$. Its characteristic polynomial $f(t) = t^2 - 3t + 2$ has two simple roots: $t = 1$ and $t = 2$. Hence, its general solution is $y_0 = C_1e^x + C_2e^{2x}$. Now, find a partial solution corresponding to the right hand side term e^x. It can be found in the form $y_1 = Axe^x$. (Note the x factor, appearing due to the fact that $t = 1$ is a root of the characteristic polynomial!) Plug in to get $2Ae^x - 3Ae^x = e^x$, whence $A = -1$ and $y_1 = -xe^x$. Next, find a partial solution corresponding to the right hand side term $\sin 2x$. It can be found in the form $y_2 = B\sin 2x + C\cos 2x$. (Note that **both** sin and cos must be present!) As before, plug in to get $(-2B + 6C)\sin 2x + (-6B - 2C)\cos 2x = \sin 2x$, whence $-2B + 6C = 1$ and $-6B - 2C = 0$. Thus, $B = -1/20$ and $C = 3/20$. Finally, the general solution is $y = y_0 + y_1 + y_2 = \boxed{C_1e^x + C_2e^{2x} - xe^x - \frac{1}{20}\sin 2x + \frac{3}{20}\cos 2x}$.

Problem 4. Find a general solution to $y'' - 2y' + y = xe^x \ln x$.

SOLUTION: First, solve the homogeneous equation $y'' - 2y' + y = 0$. Its characteristic polynomial $f(t) = t^2 - 2t + 1$ has one double root $t = 1$; hence, its general solution is $y = C_1e^x + C_2xe^x$. Now use variation of parameters: $C_1'(x)$ and $C_2'(x)$ are found from the system

$$C_1'e^x + C_2'xe^x = 0, \qquad C_1'e^x + C_2'(xe^x + e^x) = xe^x \ln x,$$

which gives $C_2' = x\ln x$ and $C_1' = -C_2'x = -x^2\ln x$. It remains to integrate:

$$C_1 = -\int x^2 \ln x\,dx = -\frac{1}{3}x^3\ln x + \int \frac{1}{3}x^3\frac{dx}{x} = -\frac{1}{3}x^3\ln x + \frac{1}{9}x^3,$$
$$C_2 = \int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{dx}{x} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2.$$

(In both the cases one can use integration by parts.) After substituting and collecting similar terms we obtain the general solution: $\boxed{y = C_1e^x + C_2xe^x + \frac{1}{6}x^3e^x\ln x - \frac{5}{36}x^3e^x}$.

Problem 5. Find a general solution to $yy'' + (y')^3 = (y')^2$.

SOLUTION: Let $y' = p(y)$. Then $y'' = pp_y'$ and the equation becomes $ypp' + p^3 = p^2$. Separate the variables and integrate: $\dfrac{dp}{p(1-p)} = \dfrac{dy}{y}$, whence $\ln\left|\dfrac{p}{1-p}\right| = \ln|y| + C$ or, resolving in p, $\dfrac{p}{1-p} = Cy$ and $p = \dfrac{Cy}{1+Cy}$. If $C = 0$, then $p = 0$ and $y = \text{const}$. Obviously, these are solutions. If $C \neq 0$, we obtain a separable equation $\dfrac{dy}{dx} = \dfrac{Cy}{1+Cy}$. Finally, the solution is: either $\boxed{y = \text{const}}$, or $\boxed{x = y + C_2\ln|y| + C_1}$ (where $C_2 = 1/C_1$).