Math 220-05, Fall 2002

Solutions to Final Exam

Problem 1. If possible, diagonalize the matrix and find an orthogonal basis in which it has diagonal form:

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}.$$

SOLUTION: The characteristic polynomial is $f_A(x) = x^3 - 9x^2 - 9x + 81$. By trial and error one finds that $x = 3$ is a root, and the division gives $f_A(x) = (x-3)(x^2 - 6x - 27)$. Hence, the two other roots are -3 and 9. All roots are real and simple, and the matrix is diagonalizable. The eigenvectors are found from the corresponding systems, and final answer is

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \text{in the basis} \qquad \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \right\}.$$

The requirement to find an **orthogonal** basis was intentionally misleading. When all eigenvalues are distinct, the basis is rigid (one can only multiply vectors by numbers); on the other hand, since the original matrix is symmetric, the basis found is **automatically** orthogonal!

Problem 2. Find A^{-1} for

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 4 & 0 & 1 & -2 \end{bmatrix}.$$

SOLUTION: I didn't mean you to use the discriminant formula for A^{-1}, though you can always use any way you prefer. Anyway, here is the answer:

$$A^{-1} = \begin{bmatrix} -4 & -3 & -2 & -1 \\ -7 & -6 & -4 & -2 \\ -10 & -8 & -6 & -3 \\ -13 & -10 & -7 & -4 \end{bmatrix}.$$

Problem 3. Evaluate

$$\begin{vmatrix} 1 & 2 & 2 & 0 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 3 & 5 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 0 & 4 \end{vmatrix}.$$

SOLUTION: The calculation is straightforward (simplifying the matrix via elementary transformations and using row/column expansion), and the correct answer is -2.

Problem 4. Let A be an $(n \times n)$-matrix. Show that, if $\det(A - \lambda I_n) = 0$ for some $\lambda \in \mathbb{R}$, then $\det(A^2 - \lambda^2 I_n) = 0$.

SOLUTION: *First approach*: Note that $A^2 - \lambda^2 I_n = (A - \lambda I_n)(A + \lambda I_n)$ (as A commutes with I_n and $I_n^2 = I_n$). Then $\det(A^2 - \lambda^2 I_n) = \det(A - \lambda I_n)\det(A + \lambda I_n) = 0$.

Second approach: The hypotheses mean that λ is an eigenvalue of A, i.e., there is a vector $\mathbf{x} \neq 0$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Then $A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, i.e., λ^2 is an eigenvalue of A^2.

Problem 5. Let P_3 be the space of polynomials of degree up to 3 with the inner product

$$(p, q) = p(1)\cdot q(1) + p'(1)\cdot q'(1) + p''(1)\cdot q''(1) + p'''(1)\cdot q'''(1)$$

and $W = \mathrm{Span}\{1, t^2\}$. Find a basis for $\mathrm{Ker}\,\mathrm{proj}_W$.

SOLUTION: First, note that $\mathrm{Ker}\,\mathrm{proj}_W = W^\perp$ (why?). Hence, it is given by the equations $(p, 1) = (p, t^2) = 0$. The first one gives $p(1) = 0$, the second one, $p(1) + 2p'(1) + 2p''(1) = 0$. In the basis $\{1, (t-1), (t-1)^2, (t-1)^3\}$ the solution is immediate: $p = \alpha(t-1)^3 + \beta[(t-1)^2 - 2(t-1)]$. Hence, one can take for a basis $\{(t-1)^3, (t-1)^2 - 2(t-1)\}$.

Remark. Writing down a formula for proj_W and then finding its kernel is a **lot** of work! A common mistake was ignoring the fact that for the formula one needs an **orthogonal** basis in W.