Math 220-04 Linear Algebra, Fall 1999

Solutions to Midterm 2

Problem 1. Let P_4 be the space of polynomials of degree $\leqslant 4$. Prove that

$$(p,q) = \int_0^2 (t-1)^2 p(t) q(t)\, dt$$

is an inner product and, given this inner product, find a basis for $\operatorname{Ker} \operatorname{proj}_W$, where $W = \operatorname{Span}\{1, t, t^2\}$.

Solution: First, let's verify that (p,q) is an inner product. The facts that it is symmetric ($(p,q) = (q,p)$) and bi-linear ($(cp,q) = c(p,q)$ for $c \in \mathbb{R}$ and $(p+q,r) = (p,r)+(q,r)$) follow immediately from the properties of integrals. Since the integral of a nonnegative function is nonnegative, one also has $(p,p) \geqslant 0$. Finally, $(p,p) = 0$ implies $p(t) \equiv 0$ **because the integrand is continuous.** (In general this is **not** true!) A common mistake was not understanding the difference between 'if' and 'only if' in the first axiom.

Since $\operatorname{Ker} \operatorname{proj}_W = W^\perp$, we do not need the projection itself. (An attempt to write down a formula for the projection leads to a lengthy calculation and causes another common mistake: using the formula for a basis that is not orthogonal.) Finally, another simplification is to pick $\{1, (t-1), (t-1)^2, (t-1)^3, (t-1)^4\}$ for a basis in P_4, as this way many pairs of basis vectors are orthogonal. (Calculation of the products of basis vectors reduces then to the integral $\int_0^2 (t-1)^m\, dt$, which equals 0 if m is odd or $2/(m+1)$ if m is even.) Thus, pick a generic polynomial $p(t) = a_0 + a_1(t-1) + a_2(t-1)^2 + a_3(t-1)^3 + a_4(t-1)^4$. The condition $(p,1) = (p,t) = (p,t^2) = 0$ gives us the system

$$\frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 = 0, \qquad \frac{2}{5}a_1 + \frac{2}{7}a_3 = 0, \qquad \frac{2}{5}a_0 + \frac{2}{7}a_2 + \frac{2}{9}a_4 = 0,$$

which, though not extremely pleasant, still can be solved to yield $a_0 = \frac{5}{21}a_4$, $a_1 = -\frac{5}{7}a_3$, $a_2 = -\frac{10}{9}a_4$. (Note that the system splits, in fact, into two, one in a_0, a_2, a_4 and one in a_1, a_3.) Thus, one can take for a basis $\boxed{\{7(t-1)^3 - 5(t-1), 63(t-1)^4 - 70(t-1)^2 + 15\}}$.

Problem 2. The inner product on $\mathbb{R}^4$ is given by $(a,b) = a_1b_1 + 2a_2b_2 + a_3b_3 + 2a_4b_4$. Use the Gram-Schmidt process to find an orthonormal basis in $W = \operatorname{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 5 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 10 \\ -5 \\ 5 \\ 11 \end{bmatrix}.$$

Solution: A **very** common mistake (-5 pts, no objections) was using the standard inner product (instead of the given one) to calculate the length of the vectors when normalizing them. The problems assumes a straightforward calculation using known formulas; it produces

$$v_1 = u_1 \begin{bmatrix} 1 \\ -2 \\ 4 \\ 0 \end{bmatrix}, \qquad v_2 = u_2 - \frac{25}{25}v_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \qquad v_3 = u_3 - \frac{50}{25}v_1 - \frac{96}{36}v_2 = \frac{1}{3}\begin{bmatrix} -8 \\ -11 \\ -9 \\ 9 \end{bmatrix},$$

and, after normalization, $\boxed{w_1 = \frac{1}{5}v_1,\ w_2 = \frac{1}{6}v_2,\ w_3 = \frac{1}{\sqrt{61}}v_3}$.

Problem 3. Given a fixed $(m \times n)$-matrix A, define a transformation $L_A\colon M_{np} \to M_{mp}$ via $L_A(X) = AX$ (where, as usual, M_{np} and M_{mp} are the spaces of $(n \times p)$- and, respectively, $(m \times p)$-matrices). Prove that $\operatorname{rk} L_A = p \operatorname{rk} A$. (*Hint*: $\operatorname{Ker} L_A$ admits an easy explicit description in terms of the null space of A.)

Solution: To find $\operatorname{Ker} L_A$ one needs to solve the matrix equation $AX = 0$. Here X, as an $(n \times p)$-matrix, can be regarded as consisting of p independent columns, $X = [\mathbf{x}_1, \ldots, \mathbf{x}_p]$, and in order to satisfy the equation each column $\mathbf{x}_i$ must be a solution to the ordinary system $A\mathbf{x}_i = 0$, i.e., $\mathbf{x}_i$ must belong to the null space of A. Thus, $\dim \operatorname{Ker} L_A = p \operatorname{nullity} A$, and $\operatorname{rk} L_A = \dim \operatorname{Im} L_A = \dim M_{np} - \dim \operatorname{Ker} L_A = np - p \operatorname{nullity} A = p(n - \operatorname{nullity} A) = p \operatorname{rk} A$.

Problem 4. Find the polynomial $p(t)$ of degree $\leqslant 3$ such that $p(0) = p'(0) = 0$ and the value of

$$\int_0^1 \left|p(t) - 3t - 2\right|^2 dt$$

is minimal possible.

SOLUTION: Sorry. For an aesthetic reason I wrote $p(t) - 3t - 2$ instead of $3t + 2 - p(t)$. I didn't expect such a dramatic effect … Of course, one should consider P_3 with the inner product $(p, q) = \int_0^1 pq\,dt$ and find the projection of $u = 3t + 2$ to the subspace W where p is allowed to vary. The latter is given by the conditions $p(0) = p'(0) = 0$, i.e., p has the form $at^3 + bt^2$ and $W = \operatorname{Span}\{t^2, t^3\}$. To find the projection we need to orthogonalize the basis; the Gram-Schmidt yields $v_1 = t^2$ and $v_2 = t^3 - \frac{5}{6}t^2$. A direct calculation gives then $(u, v_1) = \frac{17}{12}$, $(v_1, v_1) = \frac{1}{5}$ and $(u, v_2) = -\frac{29}{360}$, $(v_2, v_2) = \frac{1}{36\cdot 7}$. (By the way, it may be a good practice to not multiply right away, as this way you can easier see cancellations in your further calculation.) Finally, we have $p = \operatorname{proj}_W u = \frac{85}{12}t^2 - \frac{203}{10}(t^3 - \frac{5}{6}t^2) = \boxed{24t^2 - \frac{203}{10}t^3}$.

Problem 5. The transformation $L\colon P_4 \to P_2$ is given by

$$p(t) \mapsto p(-2) + (t-1)p'(-2) + (t-1)^2 p''(-2).$$

Find its rank and bases for $\operatorname{Ker} L$ and $\operatorname{Im} L$.

SOLUTION: Take $\{1, (t+2), (t+2)^2, (t+2)^3, (t+2)^4\}$ for a basis in P_4 and let $p = a_0 + a_1(t+2) + a_2(t+2)^2 + a_3(t+2)^3 + a_4(t+2)^4$. Since 1, $(t-1)$, and $(t-1)^2$ are linearly independent, the condition $L[p] = 0$ for $\operatorname{Ker} L$ turns into $p(-2) = p'(-2) = p''(-2) = 0$, i.e., $a_0 = a_1 = a_2 = 0$. Thus, $\boxed{(t+2)^3 \text{ and } (t+2)^4}$ form a basis in $\operatorname{Ker} L$. (I don't want to even discuss the calculation in the standard basis; it must be horrible, and this was deliberate.) Since $\dim \operatorname{Ker} L = 2$ and $\dim P_4 = 5$, one has $\boxed{\operatorname{rk} L = \dim \operatorname{Im} L = 5 - 2 = 3}$, and, as also $\dim P_2 = 3$, the image of L coincides with P_2 and any basis of P_2, say, $\boxed{\{1, t, t^2\}}$ will do. (By the way, there is no such thing as $\operatorname{rk} \operatorname{Ker} L$ or $\operatorname{rk} \operatorname{Im} L$; there are $\dim \operatorname{Ker} L$, $\dim \operatorname{Im} L$, and $\operatorname{rk} L$; the latter, by definition, is $\dim \operatorname{Im} L$ or, else, the rank of the matrix representing L with respect to any pair of bases.)