Homework #5
(due on 1/6)

1: Prove that, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and $a_n \leqslant c_n \leqslant b_n$ for all n, then $\sum_{n=1}^{\infty} c_n$ also converges. (*Warning*: we do not assume that the terms are positive!) What can be said about $\sum_{n=1}^{\infty} c_n$ if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge?

2: Show that, if $\lim_{n\to\infty} na_n$ exists and is not equal to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges. Show that, if a_n is a decreasing sequence and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Does the latter statement hold without the assumption that a_n is decreasing?

3: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}, \quad \sum_{n=1}^{\infty} (\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})\dots(\sqrt{2}-\sqrt[2n+1]{2}), \quad \sum_{n=2}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}.$$

4: Test for absolute/conditional convergence:

$$\sum_{n=2}^{\infty} \sin\frac{n\pi}{12}(\ln n)^{-1}, \quad \sum_{n=2}^{\infty} \ln\left(1+\frac{(-1)^n}{n}\right), \quad \sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n^p},\ p \in \mathbb{R},$$

where $[x]$ stands for the integral part of $x \in \mathbb{R}$.

5: Assume that all functions $u_n(x)$, $n \in \mathbb{N}$, are continuous on a segment $[a,b]$ and the limit $f(x) := \lim_{n\to\infty} u_n(x)$ exists and is also continuous. Is the convergence of the sequence $u_n(x)$ uniform on $[a,b]$?

Homework #4
(due on 12/9)

1: ('Bezout's theorem' for arbitrary functions). Assume that $f(x)$ is continuous in a neighborhood of $a \in \mathbb{R}$. Under what conditions on f does $(x-a)$ 'divide' f, *i.e.*, the function $g(x) := f(x)/(x-a)$ has a continuous extension through a? What is $g(a)$?

2: Under what conditions on $m, n \in \mathbb{R}$ the function

$$f(x) = \begin{cases} |x|^m \sin\dfrac{1}{|x|^n}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

(1) is continuous at 0?
(2) is differentiable at 0?
(3) has bounded derivative in a neighborhood of 0?
(4) is continuously differentiable at 0?

3: Prove that the *Legendre polynomial* $P_m(x) := \dfrac{1}{2^m m!}\dfrac{d^m}{dx^m}(x^2-1)^m$ satisfies the differential equation $(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$.

4: Prove that, if all roots of a polynomial $f(x)$ are real and belong to a segment $[a, b]$, then all roots of $f'(x)$ are also real and belong to $[a, b]$. (*Attention*: do not forget that a polynomial may have multiple roots!) Prove that all roots of the Legendre polynomial $P_m(x)$ are real and belong to $(-1, 1)$.

5: Prove that, if $f(x)$ is differentiable on (a, b) and $f'(x)$ is bounded, then $f(x)$ is also bounded. Does the converse hold?

Homework #3
(due on 11/1)

Let us discuss limits and continuity. In all problems, f is a function defined on a certain domain $D \subset \mathbb{R}$. When speaking about limits, we assume that a is an accumulation point of D.

1: True or false: $\lim_{x\to a} f(x) = L$ if and only if, for any *increasing* sequence $x_n \in D$, $x_n \neq a$, $x_n \to a$ one has $f(x_n) \to L$?

2: True or false: $\lim_{x\to a} f(x) = L$ if and only if, for any *monotonous* sequence $x_n \in D$, $x_n \neq a$, $x_n \to a$ one has $f(x_n) \to L$?

3: Let us try to get rid of L in the definition of limit. True or false: f has a limit at $x \to a$ if and only if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $x_1, x_2 \in D \smallsetminus \{a\}$, $|x_1 - x_2| < \delta$, and $|x_i - a| < \delta$ for $i = 1, 2$, one has $|f(x_1) - f(x_2)| < \varepsilon$?

4: What should be changed in #3 to obtain a criterion for 'f is continuous at a'?

5: True or false: f is continuous on D if and only if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $x_1, x_2 \in D$ and $|x_1 - x_2| < \delta$, one has $|f(x_1) - f(x_2)| < \varepsilon$?

Now, think again and explain what it is that you have just defined.

Homework #2
(due on 10/18)

1: Compute $\sqrt{2}^{\sqrt{2}^{\sqrt{2}\cdots}}$. *Hint*: consider the sequence a_n defined recursively *via* $a_1 = \sqrt{2}$, $a_n = \sqrt{2}^{a_{n-1}}$ for $n \geqslant 2$. Find its limit, if exists.

2: Prove that the set of real numbers is unique up to isomorphism. *Hint*: assume that there are two sets $\mathbb{R}_1$, $\mathbb{R}_2$ satisfying all axioms and show that there is a *unique* map $f\colon \mathbb{R}_1 \to \mathbb{R}_2$ preserving the arithmetical operations and order. To this end, try to construct f 'algebraically' as far as possible (done in class) and then extend it by continuity (explain what this means and why it works). Then use 'general nonsense' to show that such a map f is necessarily an isomorphism (here, the uniqueness is a hint).

Homework #1
(due on 10/11)

38/7: (a) Let a be a fixed real number, and define $x_n := a$ for $n \in \mathbb{N}$. Prove that the 'constant' sequence x_n converges.

(b) What does x_n converge to?

38/8: Suppose that x_n is a sequence in $\mathbb{R}$. Prove that x_n converges to a if and only if **every** subsequence of x_n converges to a.

44/4: Suppose that $x \in \mathbb{R}$, $x_n \geqslant 0$, and $x_n \to x$ as $n \to \infty$. Prove (without using the continuity of $\sqrt{x}$) that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

44/5: Prove that, given $x \in \mathbb{R}$, there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$.

44/9: Interpret a decimal expansion $0.a_1a_2\ldots$ as

$$0.a_1a_2\ldots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k}.$$

Prove that (a) $0.5 = 049999\ldots$ and (b) $1 = 0.9999\ldots$.