Uniformization of Riemann surfaces

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Contents

Throughout these notes, I stands for the unit segment $[0,1] \subset \mathbb{R}$; this is a standard convention in homotopy theory.

Most proof in these notes are straightforward and left as exercises.

1 The fundamental group

1.1 Paths and homotopies

Let X be a topological space. (You can think of a metric space, Riemann surface, open subset of $\mathbb{C}$, *etc*.) A *path* in X is a continuous map $\gamma\colon I \to X$. A *homotopy* of paths is a continuous map $h\colon I \times I \to X$; intuitively, this is a continuous 1-parameter family of paths $h_s(t) := h(t,s)$. The notion of homotopy extends to any maps $Y \mapsto X$, a homotopy being a continuous map $Y \times I \to X$. Occasionally, a subspace $B \subset Y$ is fixed and it is required that $h(y,s) = h(y,0)$ for all $s \in I$ and $y \in B$, *i.e.*, that all points of B should stay put during the deformation.

Convention 1.1. Free homotopies of paths are not very interesting, as any path would be homotopic to constant. Therefore, from now on we adopt the following convention: unless stated otherwise, a path homotopy $h\colon I\times I \to X$ *must preserve the endpoints*, so that $h(0,s) = \mathrm{const}$ and $h(1,s) = \mathrm{const}$.

A path homotopy $h\colon I \times I \to X$ is a homotopy *between* the paths

$$\alpha := h_0\colon t \mapsto h(t,0) \quad \text{and} \quad \beta := h_1\colon t \mapsto h(t,1),$$

and we say that α and β are *connected* by h. Two paths α, β connected by a path homotopy are said to be *homotopic*, $\alpha \sim \beta$.

Proposition 1.2. *"Homotopic" is an equivalence relation,* i.e.*, it is reflexive, symmetric, and transitive. Therefore, all paths in X split into* homotopy classes; *the homotopy class of γ is denoted by* $[\gamma]$. ◁

By Convention 1.1, the endpoints $\gamma(0)$, $\gamma(1)$ are the same within a class $[\gamma]$.

Given two paths $\alpha, \beta\colon I \to X$ *such that* $\alpha(1) = \beta(0)$, their *product* is the path

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in [0,1/2], \\ \beta(2t-1), & \text{if } t \in [1/2,1]. \end{cases}$$

The *inverse* of α is the path

$$\alpha^{-1}(t) := \alpha(1-t).$$

The *constant paths* are

$$e_x(t) := x = \text{const}, \quad x \in X.$$

The next few statement are proved by constructing appropriate piecewise linear functions $I \times I \to I = I \times 0$. Note that the "homotopic" in the statements is essential: none of the conclusions holds for individual paths. (Explain!)

Proposition 1.3 (associativity)**.** *One has* $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$ *provided that one of the sides is defined* (*then so is the other*)*.*

Proposition 1.4 (identity)**.** *One has* $e_{\alpha(0)} \cdot \alpha \sim \alpha \sim \alpha \cdot e_{\alpha(1)}$*.*

Proposition 1.5 (inverse)**.** *One has* $\alpha \cdot \alpha^{-1} \sim e_{\alpha(0)}$ *and* $\alpha^{-1} \cdot \alpha \sim e_{\alpha(1)}$*.*

Proposition 1.6 (homotopy invariance)**.** *If* $\alpha \sim \alpha'$, $\beta \sim \beta'$, *then* $\alpha \cdot \beta \sim \alpha' \cdot \beta'$*.*

1.2 The fundamental group

A *loop* is a path γ such that $\gamma(0) = \gamma(1)$; we will also say that γ is a *loop at* $x_0 := \gamma(0) = \gamma(1)$. From now on we consider *based* (or *pointed*) topological spaces, *i.e.*, pairs (X, x_0), where $x_0 \in X$ is a distinguished *basepoint*. Note that any two loops at x_0 can be multiplied, the result being a loop at x_0. Hence, the following definition makes sense.

Definition 1.7. The *fundamental group* $\pi_1(X, x_0)$ is the set of (path) homotopy classes of loops at x_0, regarded as a group (see Propositions 1.3–1.6) with respect to the path product. Typically, this group is *non-abelian*.

Since the fundamental group is made out of loops (a special case of paths), it contains absolutely no information about the path components of X other than that containing x_0. Therefore, from now on we consider only path connected topological spaces.

Example 1.8. Recall that we have vaguely defined a simply connected space as one where each simple closed curve is contractible. More precisely, for a path connected space X, the following statements are equivalent:

1. $\pi_1(X, x_0) = 1$ for any point $x_0 \in X$;
2. any map $S^1 \to X$ is (freely) homotopic to a constant map;

3. any map $S^1 = \partial D^2 \to X$ extends to $D^2 \to X$.

These equivalent properties constitute the simple connectedness of X. Their equivalence is essentially a tautology; still, some work needs to be done (like explaining that loops are the same as maps $S^2 \to X$ or observing that I^2, with or without part of the boundary contracted to a point, is the same as D^2).

Thus, $\pi_1(X)$ measures the extent to which X is not simply connected.

Example 1.9. One has $\pi_1(S^1) = \pi_1(\mathbb{C} \smallsetminus 0) = \mathbb{Z}$. (Later, in §1.3, we will see that, *as an abstract group*, π_1 is independent of the base point.) The only homotopy invariant of a loop γ is the "number of times" it wraps about the circle or its "index" with respect to 0:

$$\operatorname{ind}_0 \gamma := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

(Of course, the latter formula is an overkill.) A rigorous proof will be given later, see Corollary 2.18.

Example 1.10. The group $\pi_1(\mathbb{C} \smallsetminus \pm 1)$ is the free (non-abelian) group on two generators, *e.g.*, the classes of the circles $|z \pm 1| = 1$. We will not prove this fact.

The fundamental group is a functor: a continuous map $f\colon (X, x_0) \to (Y, y_0)$ (this notation assumes that $f(x_0) = y_0$) induces a homomorphism

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha],$$

so that

$$\mathrm{id}_* = \mathrm{id} \quad \text{and} \quad (f \circ g)_* = f_* \circ g_*.$$

1.3 The translation homomorphism

We have to keep track of the basepoint in order to have π_1 as a functor. However, as an abstract group, $\pi_1(X, x_0)$ does not depend on x_0 (provided that this point is chosen within the same path component of X). More precisely, a path $\gamma\colon I \to X$ gives rise to the *translation homomorphism*

$$T_\gamma\colon \pi_1(X, \gamma(0)) \to \pi_1(X, \gamma(1)), \quad [\alpha] \mapsto [\gamma^{-1}\alpha\gamma].$$

Using Propositions 1.3–1.6, one can easily show that T_γ is well defined and is indeed a homomorphism. Furthermore, one has

$$T_{\gamma\cdot\delta} = T_\delta \circ T_\gamma, \quad T_e = \mathrm{id}\,.$$

It follows that all T_γ are, in fact, isomorphisms: $T_\gamma^{-1} = T_{\gamma^{-1}}$. Besides, T_γ depends on the path homotopy class of γ only. If γ itself is a loop at x_0, then T_γ is the inner automorphism $[\alpha] \mapsto [\gamma]^{-1} \cdot [\alpha] \cdot [\gamma]$. Thus, if $\pi_1(X)$ happens to be abelian, all groups $\pi_1(X, x)$ are *canonically* isomorphic; otherwise, the isomorphisms are only canonical up to an inner automorphism.

2 Coverings

The theory of covering spaces is one of those rare cases where a homotopy-theoretic invariant, *viz.* the fundamental group, gives us a complete solution to a geometric problem.

2.1 Covering spaces

A *covering* (do not mix with open coverings considered, *e.g.*, in the definition of compactness) of a topological space B is a map $p\colon X \to B$ such that each point $b \in B$ has a *well-covered* neighborhood, *i.e.*, a neighborhood $U \ni b$ whose pull-back $p^{-1}(U)$ is a disjoint union $\bigsqcup_\alpha U_\alpha$ of "copies" of U, so that each restriction $p\colon U_\alpha \to U$ is a homeomorphism. The terminology is as follows:

- B is the *base* of the covering,
- X is the *total* or *covering* space, and
- p is the *covering projection*.

The pull-backs $p^{-1}(b)$, $b \in B$, are called *fibers*.

We will mainly confine ourselves with the coverings in the *strict sense*, *i.e.*, we will assume that both the base B and covering space X are path connected. In this case, the *degree* $\deg p := |p^{-1}(b)|$ (which may be finite or infinite) is independent of $b \in B$ (as it is obviously locally constant; immediately from the definition).

Example 2.1. For any space B, the identity $\mathrm{id}\colon B \to B$ is a *trivial* covering, of degree 1.

Example 2.2. Let $S^1 := \{|z| = 1|\} \subset \mathbb{C}$ be the unit circle. Then the restriction $n\colon S^1 \to S^1$ of the map $z \mapsto z^n$ is a covering of degree n. The map $u\colon \mathbb{R} \to S^1$, $t \mapsto e^{it}$, is a covering of degree ∞. The latter map can also be visualized using the *helix*, *i.e.*, the parametrized curve

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad t \in \mathbb{R}$$

in $\mathbb{R}^3$. The helix itself is homeomorphic to $\mathbb{R}$ (*e.g.*, the projection to the z-axis), whereas its projection to the xy-plane is a covering of the unit circle $\{x^2+y^2=1\}$. It is like an infinite spring squeezed to a wire ring.

Example 2.3. Likewise, the map $z \mapsto z^n$ is a covering $n\colon \mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0$ of degree n, whereas $z \mapsto e^z$ is a degree ∞ covering $\exp\colon \mathbb{C} \to \mathbb{C} \smallsetminus 0$. In both cases, for a well-covered neighborhood of a point $b \in \mathbb{C} \smallsetminus 0$ one can take the sector

$$\{\alpha - \pi < \arg w < \alpha + \pi\},$$

where α is any fixed value of $\arg b$.

Warning 2.4. Coverings are not to be mixed with local homeo-(diffeo-, or such) morphisms, which are "good" locally in the domain X rather than target B (which is a much weaker condition). In Example 2.2, the restriction of the projection to any open interval of the helix would be a local diffeomorphism, but not a covering: the "images" of the two endpoints of the interval would have no well-covered neighborhoods. Similarly (*cf.* Example 2.3), any holomorphic function $f\colon U \to \mathbb{C}$ with nowhere vanishing derivative is a local diffeomorphism, but typically not a covering.

2.2 Path and homotopy lifting

The next two statements are the main technical tool used in the study of coverings.

Lemma 2.5 (path lifting)**.** *Given a covering $p\colon X \to B$, a path $\gamma\colon I \to B$ starting at $b_0 := \gamma(0)$, and a point $x_0 \in p^{-1}(b_0)$, there is a unique* lift *$\tilde{\gamma}\colon I \to X$ of γ starting at x_0,* i.e.*, path $\tilde{\gamma}\colon I \to X$ such that $\tilde{\gamma}(0) = x_0$ and $p \circ \tilde{\gamma} = \gamma$.*

Proof. Since I is compact, by *Lebesgue lemma* applied to the open covering (in the old sense!) $\gamma^{-1}(U)$ by the well-covered neighborhood, there is a subdivision $t_0 < t_1 < \ldots < t_n$, $t_i := i/n$, $i = 0, \ldots, n$, such that each image $\gamma[t_{i-1}, t_i]$ fits into a well-covered neighborhood. Then the lift is constructed and its uniqueness is proved step-by-step, by induction in i. At each step, the image $\gamma[t_{i-1}, t_i]$ is in a well-covered neighborhood U and $\tilde{\gamma}(t_{i-1}) \in U_\alpha$ (see the definition) is already defined; hence, we have no choice but to extend $\tilde{\gamma}$ by "the same path" γ in the identical copy U_α of U. □

Lemma 2.6 (homotopy lifting)**.** *Given a covering $p\colon X \to B$, a* (*free*) *homotopy $h\colon I^2 \to B$, and a lift $\tilde{\gamma}\colon I \to X$ of the path $\gamma(t) := h(t, 0)$, there is a unique lift $\tilde{h}\colon I^2 \to X$ of h such that $\tilde{h}(t, 0) = \tilde{\gamma}(t)$.*

The proof repeats that of Lemma 2.5: we subdivide I^2 into squares S_{ij} of size $(1/n) \times (1/n)$, ordered lexicographically, and construct the lift square by square, extending what is already given on (part of) the boundary.

Corollary 2.7 (of Lemma 2.6)**.** *Let $p\colon X \to B$ be a covering. Pick a point $x_0 \in X$ and denote $b_0 := p(x_0) \in B$. Then the induced homomorphism*

$$p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$$

is a monomorphism, so that $\pi_1(X, x_0)$ is identified with a subgroup of $\pi_1(B, b_0)$.

The subgroup $\pi_1(X, x_0) \subset \pi_1(B, b_0)$ is called the *group of the covering* p; its elements are the loops in B that lift to *loops* (rather than just paths) in X. (Note that Lemma 2.5 does *not* guarantee that a loop lifts to a loop!) Strictly speaking, this subgroup is defined for a based covering of a based space, *i.e.*, the basepoints x_0 and b_0 should be fixed. The covering p is called *regular*, or *Galois*, if this subgroup is normal.

Corollary 2.8 (of Lemmas 2.5 and 2.6)**.** *Under the assumptions of Corollary 2.7, there is a canonical bijection $p^{-1}(b_0) = \pi_1(X, x_0)\backslash\pi_1(B, b_0)$.*

Proof. The construction is very typical for the theory of covering spaces. Given $x \in p^{-1}(b_0)$, consider a path $\tilde{\gamma}\colon I \to X$ connecting x_0 to x. (Recall that all spaces are assumed path connected.) Then $p \circ \tilde{\gamma}$ is a loop in B, and we take its coset for the image of x. Conversely, given a loop γ at b_0, lift it to a path $\tilde{\gamma}$ in X starting from x_0 and let $x := \tilde{\gamma}(1)$. The result is homotopy invariant. Furthermore, since any loop $\alpha \in \pi_1(X, x_0)$ lifts to a *loop* $\tilde{\alpha}$, the lift of $\alpha \cdot \gamma$ is $\tilde{\alpha} \cdot \tilde{\gamma}$, which has the same endpoint $\tilde{\gamma}(1)$. □

2.3 Covering maps

Let $p_i\colon X_i \to B_i$, $i = 1, 2$, be two coverings. A *covering map* (over a map $f\colon B_1 \to B_2$) is a map $\tilde{f}\colon X_1 \to X_2$ which makes the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\tilde{f}} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

commute: $p_2 \circ \tilde{f} = f \circ p_1$. Below, we mainly stick to the case where $B_1 = B_2 = B$ and $f = \mathrm{id}\colon B \to B$. In this case, $\tilde{f}$ is itself a covering. (Explain!)

Example 2.9 (*cf.* Example 2.2)**.** We have covering maps

$$\begin{array}{ccc} S^1 & \xrightarrow{\;n\;} & S^1 \\ {\scriptstyle nm}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ S^1 & = & S^1, \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{\;\tilde{u}\;} & S^1 \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ S^1 & = & S^1, \end{array}$$

where $\tilde{u}\colon t \mapsto \exp(it/m)$. There are similar diagrams involving the coverings in Example 2.3.

For the sake of simplicity, in the next theorem we confine ourselves to based coverings of based topological spaces. The statement can be rephrased for the general case, but one would have to speak about conjugacy classes of subgroups, subconjugate groups, *etc*; this is left as an exercise.

Theorem 2.10. *Let* $p_i\colon (X_i, x_i) \to (B, b_0)$, $i = 1, 2$, *be two based coverings, so that* $p_i(x_i) = b_0$. *Then, a covering map* $\tilde{f}\colon (X_1, x_1) \to (X_2, x_2)$ (*over the identity of* B), $x_1 \mapsto x_2$, *exists if and only if* $\pi_1(X_1, x_1) \subset \pi_1(X_2, x_2)$.

Furthermore, if exists, such a map is unique.

Proof. The necessity follows from the functoriality of π_1. For the sufficiency, we explain the construction of $\tilde{f}$, leaving technical details as an exercise. (This construction is also very common in the theory.) By the assumption, we have $\tilde{f}(x_1) = x_2$. Given $x \in X_1$, find a path $\gamma_1\colon I \to X_1$ from x_1 to x, project it to the path $\gamma := p_1 \circ \gamma_1$ in B, and lift the latter to a path γ_2 in X_2 starting from x_2. Then, let $\tilde{f}(x) = \gamma_2(1)$. The assumption that $\pi_1(X_1, x_1) \subset \pi_1(X_2, x_2)$ (and, eventually, Lemma 2.6) make sure that this value is independent of the choice of γ_1. □

Corollary 2.11. *Two based coverings* $p_i\colon (X_i, x_i) \to (B, b_0)$ *are isomorphic* (*in the category of based coverings*) *if and only if* $\pi_1(X_1, x_1) = \pi_1(X_2, x_2)$ *is the same subgroup of* $\pi_1(B, b_0)$.

2.4 The universal covering

Due to Theorem 2.10 and Corollary 2.11, it appears that based coverings of a based topological space (B, b_0) are essentially the same as subgroups of $\pi_1(B, b_0)$, except that we do not know yet if any subgroup corresponds to a covering. The "smallest" covering, corresponding to the largest subgroup $\pi_1(B)$ itself, is the trivial covering $\mathrm{id}\colon B \to B$ (see Example 2.1). On the other hand, Theorem 2.10 suggests that the "largest" covering of B, covering all others, must be the one corresponding to the smallest subgroup $\pi_1(X, x) = \{1\}$; in other words, X must be simply connected, *cf.* Example 1.8. If exists, such a covering would be unique up to isomorphism by Corollary 2.11.

Definition 2.12. A covering $p\colon X \to B$ is called the *universal covering* of B if X is simply connected, $\pi_1(X) = \{1\}$, *cf.* Example 1.8.

Example 2.13. The coverings $u\colon \mathbb{R} \to S^1$ in Example 2.2 and $\exp\colon \mathbb{C} \to \mathbb{C} \smallsetminus 0$ in Example 2.3 are universal.

Theorem 2.14. *Any "good' topological space admits a universal covering.*

Hint for the proof. As a set, the total space X is the set of homotopy classes of paths starting at a fixed point $b_0 \in B$, and the covering map is $[\gamma] \mapsto \gamma(1)$. Then, one should work hard to define the topology on X and check that this is indeed the universal covering. The standard "good" conditions are that B should be

1. connected,
2. *locally path connected*, *i.e.*, any neighborhood U of any point $b \in B$ has a path connected subneighborhood $V \ni b$, and
3. *micro simply connected*, *i.e.*, any neighborhood U of any point $b \in B$ has a subneighborhood $V \ni b$ such that the homomorphism $\pi_1(V, b) \to \pi_1(U, b)$ induced by the inclusion $V \hookrightarrow U$ is trivial.

Further details can be found in any reasonable textbook in homotopy theory. □

2.5 Deck translations

A *deck translation* (*deck transformation*, or *covering transformation*) of a covering $p\colon X \to B$ is an auto-covering map $X \to X$ over the identity of B. Using a generalization of Theorem 2.10, one can easily show that deck translations constitute a group, which we denote by $\operatorname{Aut}(X, B)$. The simplest case is that of Galois coverings (use path/homotopy lifting lemmas for the proof).

Theorem 2.15. *If $p\colon (X, x_0) \to (B, b_0)$ is a Galois covering, then there is a canonical isomorphism $\operatorname{Aut}(X, B) = \pi_1(B, b_0)/\pi_1(X, x_0)$. Furthermore, this group acts simply transitively on each fiber and one has $B \cong X/\operatorname{Aut}(X, B)$.*

Not every group action can be realised as an action by deck translations of a covering. A more or less literal paraphrase of the definition boils down to the following description: a group action $G \times X \to X$ on a topological space X is a *covering space action* if each point $x \in X$ has a neighborhood $U \ni x$ such that all images gU, $g \in G$, are pairwise disjoint. If this is the case, the quotient projection $X \to X/G$ is a Galois covering (the image of each U as above being a well-covered neighborhood) and G is its deck translation group.

Corollary 2.16. *The deck translation group of the universal covering $X \to B$ is isomorphic to $\pi_1(B)$.*

Corollary 2.17. *If B admits a universal covering $X \to B$, it admits a covering corresponding to any subgroup $G \subset \pi_1(B)$.*

Proof. If $\pi_1(B) \times X \to X$ is a covering space action, so is the action of any subgroup $G \subset \pi_1(B)$; then, $X/G \to B = X/\pi_1(B)$ is the desired covering. ☐

Corollary 2.18 (of Corollary 2.16)**.** *One has $\pi_1(S^1) = \pi_1(\mathbb{C} \setminus 0) = \mathbb{Z}$.*

Proof. We compute the deck translations of the universal coverings $u\colon \mathbb{R} \to S^1$ (see Example 2.2) and $\exp\colon \mathbb{C} \to \mathbb{C} \setminus 0$ (see Example 2.3); they are the affine translations $t \mapsto t + 2\pi n$ and $z \mapsto z + 2\pi i n$, $n \in \mathbb{Z}$, respectively. ☐

3 Uniformization

Let S be a Riemann surface, which we assume connected. Clearly, any covering space $X \to S$ is canonically a Riemann surface. Indeed, each point $s \in S$ has a holomorphic chart $\varphi\colon U \to V \subset \mathbb{C}$ and, shrinking U if necessary, we can assume it well covered, so that $p^{-1}(U) = \bigsqcup_\alpha U_\alpha$. Then, the restrictions $\varphi \circ p\colon U_\alpha \to V$ can be declared holomorphic charts about the pull-backs of s. In fact, this analytic structure on X is uniquely determined by the requirement that p be holomorphic.

The same argument shows that, conversely, given a Riemann surface X and a subgroup $G \subset \operatorname{Aut} X$ (holomorphic automorphisms) such that the restricted action $G \times X \to X$ is a covering space action, then the quotient X/G is a Riemann surface and the covering projection $X \to X/G$ is holomorphic.

The following statement is immediate from the construction.

Proposition 3.1. *Any covering map $\tilde{f}\colon X_1 \to X_2$ over the identity* $\operatorname{id}\colon S \to S$ *(or, more generally, holomorphic self-map $f\colon S \to S$) is holomorphic.*

Obviously, any Riemann surface S satisfies the hypotheses of Theorem 2.14 and, hence, admits a universal covering $X \to S$. The conformal type of X is uniquely determined by that of S (*cf.* Proposition 3.1). On the other hand, since

X is simply connected, it must be one of $\mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D} := \{|z| < 1\}$. (We have partially proved this fact in the homeworks.) Combining all these observations, we obtain the following theorem.

Theorem 3.2 (uniformization)**.** *Any Riemann surface S can be represented as X/G, where $X \cong \mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$ and $G \subset \operatorname{Aut} X$ is a subgroup such that the induced action $G \times X \to X$ is a covering space action. The surface X and the $(\operatorname{Aut} X)$-conjugacy class of $G \subset \operatorname{Aut} X$ are uniquely determined by S.*

Remark 3.3. In the particular case of Riemann surfaces, the condition on the subgroup $G \subset \operatorname{Aut} X$ in Theorem 3.2 can be restated as follows:

1. each element $1 \neq g \in G$ is fixed point free, and
2. each point $x \in X$ has a neighborhood U such that $U \cap Gx = \{x\}$.

The latter condition implies that $G \subset \operatorname{Aut} X$ is discrete. Probably, the easiest way to prove the sufficiency of these condition is using a metric on X invariant under (a relevant subgroup of) $\operatorname{Aut} X$, *cf.* §3.1–§3.3 below.

In order to classify the simplest Riemann surfaces that are not simply connected (most notably, those with $\pi_1(S) = \mathbb{Z}$), we briefly consider the three cases $X = \mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$ in Theorem 3.2.

3.1 The case $X = \mathbb{P}^1$

The only covering $\mathbb{P}^1 \to S$ is the trivial one, $\operatorname{id}\colon \mathbb{P}^1 \to \mathbb{P}^1$. Indeed, $\operatorname{Aut} \mathbb{P}^1$ is the Möbius group and we know that each Möbius transformation has fixed points, *cf.* Remark 3.3(1).

Remark 3.4. There is a deeper, topological reason why the only coverings with $\mathbb{P}^1$ as the total space are $\operatorname{id}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and the covering $\mathbb{P}^1 \to \mathbb{P}^2_{\mathbb{R}} := S^2/\{\pm 1\}$ of degree 2. The argument uses compactness (each covering must be finite), Euler characteristic, and the so-called *Riemann–Hurwitz formula*. The *real projective plane* $\mathbb{P}^2_{\mathbb{R}}$ admits no analytic structure as it is non-orientable.

3.2 The case $X = \mathbb{C}$

The group $\operatorname{Aut} \mathbb{C}$ consists of the affine linear transformations $z \mapsto az + b$, which are fixed point free if and only if $a = 1$ (and $b \neq 0$). Thus, in this case, in Theorem 3.2 we can replace $\operatorname{Aut} \mathbb{C}$ with $\mathbb{C}$, acting on itself by translations.

Lemma 3.5. *A subgroup $G := \mathbb{Z}^m \subset \mathbb{R}^n$ is discrete if and only if its $\mathbb{R}$-span has dimension m (equal to the $\mathbb{Z}$-rank).*

Hint for the proof. Crucial is the following simple number-theoretic observation: given $t \in \mathbb{R}$, the subgroup $G := \{mt + n \,|\, m, n \in \mathbb{Z}\}$ spanned by t and 1 is $\mathbb{Z}$ if $t \in \mathbb{Q}$ or dense in $\mathbb{R}$ if t is irrational. The latter, in turn, is based on the less known fact that, for each $n > 0$, there is a pair $p_n, q_n \in \mathbb{Z}$, $q_n > n$, such that

$$\left| t - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

(These are usually constructed *via* the *Farey sequences*, see, *e.g.*, Wikipedia.) Hence, for the elements $a_n := p_n t - q_n \in G$, we have $0 < |a_n| < 1/q_n < 1/n$. (Recall that $a_n \neq 0$ as t is irrational.) Now, it is clear that the union of the subgroups $\mathbb{Z}a_n \subset G$, $n \in \mathbb{Z}^+$, is dense in $\mathbb{R}$.

The higher dimensional case is treated *via* coordinate projections: a subgroup cannot be discrete if in at least one direction the projections of its generators are incommensurable. □

Thus, we have but two possibilities: $G = \mathbb{Z}$ or $G = \mathbb{Z} \oplus \mathbb{Z}$.

Case 1: $G \cong \mathbb{Z}$. Up to conjugation in the full group $\operatorname{Aut}\mathbb{C}$, we can assume that G is generated by $1 \in \mathbb{C}$, *i.e.*, $z \mapsto z + 1$. The corresponding covering is

$$\mathbb{C} \to \mathbb{C} \smallsetminus 0, \quad z \mapsto \exp(2\pi i z).$$

(Here and below, the covering projection is found by a guesswork.)

Case 2: $G \cong \mathbb{Z} \oplus \mathbb{Z}$. Up to conjugation in $\operatorname{Aut}\mathbb{C}$, we can assume that G is generated by 1 and some $\tau \in \mathbb{H}$. Topologically, the quotient $\mathbb{C}/G$ is a torus T^2 (same as $\mathbb{C}/\mathbb{Z}^2$); however, the analytic structure on this surface depends on a continuous parameter τ. Since we can change the basis in G, this parameter is well defined up to the action of the modular group *PSL*$(2, \mathbb{Z})$; usually, it is chosen in the *fundamental domain* $\{|z| \geqslant 1,\ |\operatorname{Im} z| \leqslant 1/2\}$. (For the true uniqueness, one should also remove part of the boundary of this hyperbolic triangle.) The Riemann surfaces obtained in this way are called *elliptic curves*.

3.3 The case $X = \mathbb{D} \cong \mathbb{H}$

We have $\operatorname{Aut}\mathbb{H} = \mathit{PSL}(2,\mathbb{R})$, and an easy exercise (done in class) shows that any fixed point free element $g \in \operatorname{Aut}\mathbb{H}$ is conjugate to either $T\colon z \mapsto z+1$ or $R_\alpha\colon z \mapsto \alpha z$, $\alpha \in \mathbb{R}^+ \smallsetminus 1$. Combining with the results of §3.1 and §3.2, we conclude that the fundamental group of any Riemann surface is torsion free.

Another exercise in linear algebra shows that any element $g \in \operatorname{Aut}\mathbb{H}$ commuting with T is $z \mapsto z+\beta$, $\beta \in \mathbb{R}$, and any g commuting with R_α is R_β, $\beta \in \mathbb{R}^+$. Hence, any discrete abelian subgroup G (see Remark 3.3) lies in a 1-parameter subgroup isomorphic to $\mathbb{R}$; therefore, it can only be $\mathbb{Z}$ (*cf.* Lemma 3.5).

Case 1: $G \cong \mathbb{Z}$ generated by T. The covering projection is

$$\mathbb{C} \to \mathbb{D} \smallsetminus 0, \quad z \mapsto \exp(-2\pi i z).$$

Case 2: $G \cong \mathbb{Z}$ generated by R_α. The covering projection is

$$\mathbb{C} \to \mathbb{A}_r := \{1 < |w| < r\}, \quad z \mapsto \exp(\beta \ln z), \tag{3.6}$$

where $\beta := 2\pi i/\alpha$ and $r := \exp(\pi\beta)$.

In particular, we conclude that the annuli $\mathbb{A}_r$, $r \in (1,\infty)$, are pairwise non-isomorphic! (Besides, none of them is isomorphic to $\mathbb{C} \smallsetminus 0$ or $\mathbb{D} \smallsetminus 0$.) This gives us an example of a continuous family of pairwise distinct Riemann surfaces.

Generalizing, two annuli $\{r_i < |z| < R_i\}$, $0 < r_i < R_i < \infty$, $i = 1,2$, are isomorphic if and only if $R_1/r_1 = R_2/r_2$. Indeed, applying the linear transformation $z \mapsto z/r$, we can always assume that the inner radius equals 1.

3.4 Conclusion: Riemann surfaces with abelian fundamental group

Thus, we have obtained a complete list of Riemann surfaces with the "smallest", meaning abelian, fundamental groups:

- If $\pi_1(S) = 0$, then $S \cong \mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$; the last two are homeomorphic.
- If $\pi_1(S) = \mathbb{Z}$, then, apart from $\mathbb{C} \smallsetminus 0$ and $\mathbb{D} \smallsetminus 0$, there is a real 1-parameter family of annuli $\mathbb{A}_r$, see (3.6); all surfaces are homeomorphic.
- If $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$, then topologically S is a torus, but there is a complex 1-parameter family of pairwise distinct analytic structures.

3.5 Conclusion: Riemann surfaces by their "geometry"

Another way to classify Riemann surfaces is according to their "geometry", *i.e.*, whether they admit a (properly compatible with the analytic structure) complete Riemannian metric of constant Gaussian curvature K:

- *elliptic*, or *spherical* ($K > 0$): the Riemann sphere $\mathbb{P}^1$;
- *parabolic*, or *Euclidean* ($K = 0$): the "flat" surfaces $\mathbb{C}$, $\mathbb{C} \setminus 0$ and all elliptic curves (see Case 2 in §3.2);
- *hyperbolic* ($K < 0$): everything else.

In the last two cases, we use that fact that the groups $\mathbb{C} \subset \operatorname{Aut} \mathbb{C}$ (see §3.2) and $\operatorname{Aut} \mathbb{H}$ act by isometries of, respectively, the Euclidean metric on $\mathbb{C}$ and hyperbolic metric on $\mathbb{H}$. Hence, these metrics descend to any Riemann surface covered by $\mathbb{C}$ or $\mathbb{H}$. (Explain!)

In particular, we see that the vast majority of Riemann surfaces (all but the tori T^2 or the sphere S^2 punctured in at most 2 points) are hyperbolic (and a hyperbolic metric is essentially equivalent to an analytic structure), which makes hyperbolic geometry quite important.