A brief summary of complex analysis

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Contents

1 Glossary **2**

2 Complex integration **3**

3 Holomorphic functions **4**
3.1 Many definitions of holomorphic functions 4
3.2 Notes on path independence 5
3.3 Examples 5

4 Simple connectedness **6**

5 Power and Laurent series **8**
5.1 Basic properties 8
5.2 Examples 10

6 Properties of holomorphic functions **11**
6.1 Cauchy integral 11
6.2 Zeroes of holomorphic functions 11
6.3 Liouville's theorems 12
6.4 Uniqueness of analytic continuation 12
6.5 The maximal modulus principle 12
6.6 Schwartz' lemma 13

7 Residues and integrals **13**
7.1 Isolated singular points 13
7.2 Singular point at infinity 15
7.3 The residue theorem 16
7.4 Logarithmic residues and the argument principle 17

8 Application to real integrals **18**
8.1 Trigonometric functions over a period 18
8.2 Rational functions 18
8.3 Some transcendental functions 19
8.4 Key-hole shaped contours 21
8.5 Other contours 22
8.6 Summation of series 23

9 The Gamma-function **23**

10 Automorphisms of simply connected domains **25**
10.1 The Riemann sphere 25
10.2 The complex plane 26
10.3 The unit disk 26
10.4 Hyperbolic geometry 27

1 Glossary

These terms and notation are used throughout the paper.

A *domain* is a connected open subset $U \subset \mathbb{C}$.

A certain property P is said to hold *locally* on a domain U if each point $a \in U$ has a neighborhood in which P holds.

A *curve* is a piecewise smooth curve $C \subset \mathbb{C}$. By definition, C admits a *piecewise smooth* parameterization, *i.e.*, C is the image of a *continuous* map $\varphi\colon [a, b] \to \mathbb{C}$ such that there is a partition $a_0 := a < a_1 < a_2 < \ldots < a_n := b$ with φ continuously differentiable on each $[a_{k-1}, a_k]$, $k = 1, \ldots, n$.

A curve C is *closed* if $\varphi(a) = \varphi(b)$; it is *simple closed* if it is closed and has no self-intersections, *i.e.*, if $\varphi(t_1) \neq \varphi(t_2)$ for any pair $t_1 < t_2$ other than $t_1 = a$, $t_2 = b$. The following intuitive statement is known as *Jordan curve theorem*: *A simple closed curve divides $\mathbb{C}$ into two connected domains, of which exactly one is bounded.* The bounded domain U above is called the *interior* of the simple closed curve C (not to be mixed with the interior in the point-set topological sense); we write $C = \partial U$ and always assume that C is oriented counterclockwise. Other common names are as follows: C *bounds* U, or U *is bounded by* C, or a point $z \in U$ is *inside* C (appealing to common sense, but contradicting to set theory), or C *surrounds* a point $z \in U$.

Unless stated otherwise, we consider a complex valued function $f\colon U \to \mathbb{C}$ defined on a domain $U \subset \mathbb{C}$; the standard notation used throughout is

$$z = x + iy, \quad f(x + iy) = u(x, y) + iv(x, y). \tag{1.1}$$

We always assume f at least continuous.

2 Complex integration

Given a function $f\colon U \to \mathbb{C}$ as in (1.1) and a curve $C \in U$, we define the integral

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C u\,dx - v\,dy + i\int_C v\,dy + u\,dx.$$

Given a smooth parameterization $z(t)$ for C, the "substitution" reduces this to the usual Riemann integral

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Note that this *is* the usual Calculus 101 integral (and $z'(t)$ is the usual derivative with respect to a single real variable) of a complex valued function. Technically, this means that the real and imaginary parts are to be computed separately. For holomorphic functions, we will usually have better ways.

Here are the basic properties:

$$\int_C (\alpha f + \beta g)dz = \alpha\int_C f\,dz + \beta\int_C g\,dz, \quad \alpha, \beta = \text{const} \qquad \text{(linearity)};$$

$$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad \text{(additivity)};$$

$$\int_{-C} f(z)dz = -\int_C f(z)dz \qquad \text{(orientation)};$$

$$\left|\int_C f(z)dz\right| \leqslant \int_C |f(z)|\,ds \leqslant \max_{z\in C}|f(z)| \cdot \text{length}(C) \qquad (ML\text{-bound}).$$

The last property is often use to establish all kinds of convergence, like

$$\text{if } f_n \to f \textit{ uniformly on } C, \quad \text{then} \quad \int_C f_n(z)dz \to \int_C f(z)dz.$$

Typically, the integral depends on the integrant f *and the curve* C. We say that the integral of f is *path independent* if $\int_C f(z)dz = \int_D f(z)dz$ whenever the two curves C and D share the same starting and ending points. Note that this implies that $\int_C f(z)dz = 0$ for any *closed* curve C.

In a sense, *the* example of an integral is (for $n \in \mathbb{Z}$)

$$\int_{|z|=1} z^n dz = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

3 Holomorphic functions

This is our principal subject in complex analysis. Roughly, holomorphic are the differentiable functions; however, they have a number of other definitions.

3.1 Many definitions of holomorphic functions

A *continuous* function $f\colon U \to \mathbb{C}$ as in (1.1) is called *holomorphic* (or *analytic*) if it has any one of the following equivalent properties:

1. f has a continuous complex derivative $f'(z)$ on U;
2. the functions u, v are continuously differentiable (as real functions of two real arguments) and satisfy the *Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x. \tag{3.1}$$

3. f has complex derivatives $f^{(n)}(z)$ of all orders;
4. f is locally representable by a power series, *i.e.*, each point $c \in U$ has a neighborhood in which f is given by a power series centered at c, see §5.1;
5. *locally*, f has an anti-derivative, *i.e.*, function F such that $F' = f$;
6. the integral of f is *locally* path independent;
7. $\int_C f(z)dz = 0$ for any simple closed curve $C \subset U$ *whose interior is in* U;

8. (Morera's theorem) $\int_C f(z)dz = 0$ for any rectangle $C \subset U$ whose sides are parallel to the axes and *whose interior is in* U.

Due to (3.1), the components u, v of a holomorphic function are *harmonic*:

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0.$$

Given a harmonic function u, *locally* there exists v, unique up to constant, such that $f := u + iv$ is holomorphic.

3.2 Notes on path independence

Items 5 and 6 in §3.1 are also equivalent "globally" (*i.e.*, f has an anti-derivative on the whole domain U iff the integral of f is path independent), and both are equivalent to either of items 7, 8 with the requirement that the interior of C should be in U dropped. Throughout this section (except item 4) the word "locally" can be interpreted as "on any simply connected (see §4) domain $V \subset U$".

Practically, the local path independence means that $\int_C f(z)dz$ does not change if one varies the curve C *continuously*, *provided that* C *remains in* U and the endpoints of C stay fixed (unless C is closed). This observation should always be kept in mind when computing integrals of holomorphic functions: usually, one can replace C with a more convenient curve.

3.3 Examples

All "rules" (sum/difference/product/quotient/chain) of Calculus 101 and algebra work, and all formulas for (anti-)derivatives apply *whenever they make sense* (most notable exception being the formulas involving $\ln$ and inverse trigonometric functions, which are not quite well defined on $\mathbb{C}$). Examples of holomorphic functions are polynomials, rational functions (away from the roots of the denominator),

$$e^z := e^x(\cos y + i \sin y), \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z := \frac{e^{iz} + e^{-iz}}{2},$$

and various combinations thereof.

Examples of valid identities are

$$(z^n)' = nz^{n-1} \quad (n \in \mathbb{Z}), \qquad \int z^n dz = \frac{z^{n+1}}{n+1} \quad (n \in \mathbb{Z},\ n \neq -1)$$

(I systematically drop the integration constant in anti-derivatives),

$$(e^z)' = e^z, \qquad \int e^z dz = e^z, \qquad e^{z+w} = e^z e^w,$$
$$(\sin z)' = \cos z, \qquad \sin^2 z + \cos^2 z = 1, \qquad \sin 2z = 2 \sin z \cos z.$$

However, *extreme care should be taken* when working with functions like

$$\ln z := \ln|z| + i \arg z,$$

as well as radicals and inverse trigonometric functions (which can all be expressed in terms of $\ln$): just like $\arg z$, they are not quite well defined, even if the origin is removed. (One should make cuts to pass to a simply connected domain, see §4.) Thus, we have

$$(\ln z)' = \frac{1}{z}, \qquad \int \frac{dz}{z} = \ln z, \qquad \int \frac{dz}{z} \neq \ln|z|.$$

Here, the last formula (from Calculus 101) is hopelessly wrong, as $|\cdot|$ over $\mathbb{C}$ is much more "destructive" than over $\mathbb{R}$. The first two work *provided that z is restricted to a domain U where $\ln$ makes sense* (and a branch of $\ln$ is chosen); typically, one would take for U the plane with a ray $\arg z = \mathrm{const}$ (including 0) removed, with the effect of restricting $\arg z$ to an open interval of length 2π.

Example 3.2. There is no simple "rule" establishing the (non-)existence of an anti-derivative in more complicated situations. For example, on $U := \mathbb{C} \setminus [-i, i]$,

$$\int \frac{dz}{1+z^2} = \arctan z := \frac{1}{2i} \ln \frac{i+z}{i-z}$$

is well-defined, whereas a very similar function $z/(1+z^2)$ has no anti-derivative on U. You need to compute the integrals, see Example 4.2.

4 Simple connectedness

We are not in a position to treat this subject rigorously, so let us use our geometric intuition. A domain $U \subset \mathbb{C}$ is *simply connected* if it has no "holes". Important for us is the fact that, *in a simply connected domain U, the region bounded by a simple closed curve $C \subset U$ is also in U*; therefore, we can use Green's theorem.

Remark 4.1. In the textbook, a domain $U \subset \mathbb{C}$ is called simply connected if the complement $(\mathbb{C} \cup \infty) \smallsetminus U$ is connected. (Observe that we are speaking about the complement in the *extended* plane $\mathbb{C} \cup \infty$: this is crucial!) More generally, in this approach, "holes" can be defined as the connected components of $(\mathbb{C} \cup \infty) \smallsetminus U$ not containing ∞ (*i.e.*, all but one). Although this definition does provide a certain rigour, it is limited to open subsets of $\mathbb{C}$, is based on hard algebraic topology (the so-called *Alexander duality*), and, most importantly, it is absolutely unclear how it can be used with statements like Green's theorem!

Common examples of simply connected domains are:

- the plane $\mathbb{C}$ itself,
- the extended plane $\mathbb{C} \cup \infty$,
- the region bounded by a simple closed curve,
- the plane $\mathbb{C}$ with a number of *infinite* cuts, *i.e.*, pairwise disjoint *unbounded* rays removed.

Examples of the last option are (in the hope that the vague notation is understood)

- $\mathbb{C} \smallsetminus (-\infty, 0]$: one cut. This domain is "good" for $\ln$ and $\sqrt{\ }$; one can also consider $\mathbb{C} \smallsetminus \{\arg z = \mathrm{const}\}$;
- $\mathbb{C} \smallsetminus \big((-i\infty, -i] \cup [i, i\infty)\big)$: two cuts; "good" for the anti-derivative of the function $z/(1+z^2)$ considered in Examples 3.2 and 4.2.

Here are examples of domains that are not simply connected:

- $\mathbb{C} \smallsetminus 0$: one "hole";
- $\mathbb{C} \smallsetminus [-i, i]$ (*cf.* Examples 3.2 and 4.2): one "hole";
- an annulus $r < |z| < R$: one "hole" (assuming $r \geqslant 0$);
- $\mathbb{C} \smallsetminus \{\pm i\}$: two "holes", *cf.* Example 4.2;
- $\mathbb{C} \smallsetminus \mathbb{Z}$: infinitely many "holes".

In a simply connected domain U, one can drop the word "local" in all statements in §3.1: holomorphic functions have anti-derivatives, their integrals are path independent, for each u there is a v, *etc*. In general, the global path independence of integral of f is equivalent to the existence of an anti-derivative F:

$$\int_a^b f(z)dz = F(b) - F(a), \qquad F(z) = \int_a^z f(w)dw,$$

where $\int_a^b$ stands for the integral along *any* curve $C \subset U$ from a to b. However, in order to establish this property, for each "hole" in U one needs to compute the integral $\int_C f(z)dz$ along (any) one simple closed curve $C \subset U$ surrounding this (and only this) hole and *make sure that all these integrals vanish*. This is usually done by means of residues, see §7.

Example 4.2. The domain $U := \mathbb{C} \smallsetminus [-i, i]$, see Example 3.2, has a single hole, *viz.*, the segment $[-i, i]$. Any simple closed curve $C \subset U$ surrounding this hole contains both poles $\pm i$ inside, so that $\int_C f\,dz = 2\pi i(\operatorname{res}_i f + \operatorname{res}_{-i} f)$. We have

$$\int_C \frac{dz}{1+z^2} = 0, \qquad \int_C \frac{z\,dz}{1+z^2} \neq 0$$

(*cf.* also Example 7.7); hence, the former function has an anti-derivative, and the latter does not. On the larger domain $V := \mathbb{C} \smallsetminus \{\pm i\}$, we would have to compute two integrals, one for each of the two holes. They do not vanish (as neither do the residues) and, hence, neither function has an anti-derivative on V.

5 Power and Laurent series

This is one of the principal tools in the study of holomorphic functions.

5.1 Basic properties

A *power/Laurent series* centered at a point $c \in \mathbb{C}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-c)^n \quad \text{or} \quad \sum_{n=-\infty}^{\infty} a_n(z-c)^n, \tag{5.1}$$

respectively, where the constants $a_n \in \mathbb{C}$ are the *coefficients*. Each series (5.1) has a *radius* $0 \leqslant R \leqslant \infty$ (respectively, *radii* $0 \leqslant r \leqslant R \leqslant \infty$) *of convergence*, so that the series converges *absolutely and uniformly on compacta* on the disk/annulus

$$U := \bigl\{|z-c| < R\bigr\} \quad \text{or} \quad U := \bigl\{r < |z-c| < R\bigr\},$$

respectively and diverges outside the closure of U. (The behaviour of the series at various boundary in ∂U may vary from series to series.) Certainly, we are interested in the cases where $R > 0$ (respectively, $r < R$), so that the disk/annulus of convergence U is nonempty.

Consider a series (5.1) and let U be its region of convergence. Then:

- the sum $f(z)$ of (5.1) is holomorphic on U;
- conversely, U is the maximal disk (a maximal annulus) to which f extends holomorphically (this is how we usually find the radii of convergence);
- a power/Laurent series can be integrated and differentiated termwise; for the integration, *cf.* (2.1) and (5.2).

It follows that an *entire* (holomorphic on the whole plane $\mathbb{C}$) function admits a Taylor expansion centered at any point, and its radius of convergence is ∞.

The power/Laurent series expansion of a function f that is holomorphic on a disk/annulus U about $c \in \mathbb{C}$ is unique; its coefficients can be found *via*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-c)^{n+1}}, \tag{5.2}$$

where $C \subset U$ is a simple closed curve surrounding c. (Usually, one takes for C the circle $|z| = \rho$ for some $0 < \rho < R$ or $r < \rho < R$, respectively.) Note, though, that these formulas are not very practical; usually, one finds an expansion by manipulating (integration, differentiation, substitution, multiplication, *etc.*) a few known series. (Often, partial fractions help one convert products tu sums, which are much easier to handle.)

Example 5.3. Note also that the *Laurent* series depends on the point c *and* the chosen annulus of convergence! For example,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = -\sum_{n=-\infty}^{-1} z^n,$$

converging in $|z| < 1$ and $|z| > 1$, respectively! This is why we speak about *a* maximal annulus for Laurent series.

The coefficients of the power (Taylor) series expansion are also found *via*

$$a_n = \frac{1}{n!} f^{(n)}(c),$$

as in Calculus 101. These formulas do not apply to Laurent series, as in this case f is not even assumed defined at c.

Power/Laurent series can also be multiplied like polynomials:

$$\left(\sum_{n=-\infty}^{\infty} a_n(z-c)^n\right)\left(\sum_{n=-\infty}^{\infty} b_n(z-c)^n\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{p+q=n} a_p b_q\right)(z-c)^n.$$

If both series are power (more generally, both have but finitely many negative terms), the sums $\sum_{p+q=n} a_p b_q$ in the right hand side are finite. Otherwise, these sums are also series, which are guaranteed to converge absolutely provided that the regions of convergence of the two original series have nonempty intersection.

5.2 Examples

The following Taylor series are assumed known from Calculus 101:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (R = 1, \textit{geometric}), \tag{5.4}$$

$$(1+z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n, \quad a \in \mathbb{C} \qquad (R = 1, \textit{binomial}), \tag{5.5}$$

$$\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} z^n, \quad m \in \mathbb{Z}^+ \quad (R = 1, \text{special case}), \tag{5.6}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (R = \infty, \textit{exponential}), \tag{5.7}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \qquad (R = \infty, \text{special case}), \tag{5.8}$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \qquad (R = \infty, \text{special case}). \tag{5.9}$$

As mentioned above, other series are usually obtained from these by substitution (either simple rescaling $z \mapsto \alpha z$ or converting Taylor to Laurent *via* $z \mapsto 1/z$), integration/differentiation, linear combinations, or multiplication (if everything else fails). For example, the second series in Example 5.3 is essentially obtained from (5.4) by substituting $z \mapsto 1/z$. This would apply to any rational function: one can expand it to partial fractions (over $\mathbb{C}$, we can stick to fractions with *linear* factors of the form $1/(1-\alpha z)^m$) and use (5.6), substituting either $z \mapsto \alpha z$ or $z \mapsto 1/\alpha z$, depending on the position of the pole $1/\alpha$ with respect to the desired annulus of convergence.

Example 5.10. Occasionally, one can also use (5.2), comparing the integrals to known ones. Back to Example 5.3, both series have coefficients

$$a_n, b_n = \frac{1}{2\pi i} \int_C f_n(z)dz, \qquad f_n(z) := \frac{1}{z^{n+1}(1-z)}.$$

The only difference is whether the extra pole $z = 1$ is inside (for b_n, Laurent) or outside (for a_n, Taylor) the curve C. Since $\operatorname{res}_1 f_n = -1$ for each $n \in \mathbb{Z}$, we immediately conclude that $b_n = a_n - 1$, which agrees to the previous result. See §7 and Theorem 7.5 for the details of this computation.

6 Properties of holomorphic functions

In this section, I summarize a few miscellaneous (although important) properties of holomorphic functions which have no analogues in real analysis.

Throughout this section, $U \subset \mathbb{C}$ is a domain, *i.e.*, a connected open set.

6.1 Cauchy integral

Let $f\colon U \to \mathbb{C}$ be a holomorphic function, $z \in U$, and $C \subset U$ a simple closed curve surrounding $z \in U$ and bounding a region V *contained in* U. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z}, \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)dw}{(w-z)^{n+1}} \quad \text{for } n \geqslant 1.$$

These formulas have numerous remarkable consequences, as in general integrals behave much better than derivatives (*e.g.*, they commute with uniform limits).

Theorem 6.1. *If all functions $f_n\colon U \to \mathbb{C}$ are holomorphic and $f_n \to f$* uniformly on compacta*, then f is also holomorphic and $f_n^{(m)} \to f^{(m)}$ for each $m \geqslant 1$.*

6.2 Zeroes of holomorphic functions

In some respects, holomorphic functions are similar to polynomials; for example, if $f(a) = 0$, then f is "divisible" by $z - a$. More precisely, assume that $f\colon U \to \mathbb{C}$ is a holomorphic function, no identically 0, and $f(a) = 0$ for some $a \in U$. Then there is an integer $m \geqslant 0$ and a holomorphic function $g\colon U \to \mathbb{C}$ such that

$$f(z) = (z-a)^m g(z) \quad \text{and} \quad g(a) \neq 0.$$

This integer m is called the *multiplicity*, or *order*, of a as a zero of f; it is the order of the first non-vanishing derivative of f:

$$f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0, \quad f^{(m)}(a) \neq 0.$$

Alternatively, m is also the index of the first non-vanishing term of the Taylor expansion of f about a.

In particular, it follows that zeroes of holomorphic functions are isolated.

6.3 Liouville's theorems

Here are the numerous versions of Liouville's theorem, although I am not sure that they all bear this name. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Then:

1. if $f \to 0$ as $z \to \infty$, then f is identical 0;
2. if $|f|$ is bounded, then $f = \mathrm{const}$;
3. if $f \to \infty$ as $z \to \infty$, then f is a polynomial;
4. if $|f(z)| < C|z|^n$ for some constants C and n and all $|z| \gg 0$, then f is a polynomial of degree at most n.

Furthermore,

5. if f is a *meromorphic* (*i.e.*, holomorphic except for finitely many singularities, which are all poles) on the *extended plane* $\mathbb{C} \cup \infty$, then f is rational.

6.4 Uniqueness of analytic continuation

Roughly, holomorphic functions are very "rigid": if two such functions coincide on a sufficiently large set (in fact, a set with at least one limit point), they coincide everywhere. In particular, if a function admits (which we do not state) an analytic continuation to a larger domain, this continuation is unique.

Theorem 6.2. *Let U be a domain, $f, g\colon U \to \mathbb{C}$ two holomorphic functions, and $a_n \in U$ a sequence with pairwise distinct terms converging to a point $a \in U$. If $f(a_n) = g(a_n)$ for all n, then $f \equiv g$ on U.*

One of the applications of this theorem is the fact that any known "reasonable" identity (*e.g.*, $e^{x+y} = e^x e^y$) that holds for all real arguments of some known real functions holds also for all complex values of the arguments.

6.5 The maximal modulus principle

If $f\colon U \to \mathbb{C}$ is a non-constant holomorphic function, then the function $|f(z)|$ has no local maxima. Often, this is restated as follows: if $K \subset U$ is compact, then the restriction of $|f(z)|$ to K takes its maximal value (which we know it does) on the boundary ∂K.

Applying this result to $1/f$, one obtains the *minimum modulus principle*: If $f\colon U \to \mathbb{C}$ is a non-constant holomorphic function, then the function $|f(z)|$ has no local minima *other than the zeroes of f*.

Here is a typical and quite shocking application, used in the proof of the next theorem. Let $K \subset U$ be a compact, and assume that there is an *inner* point $a \in K$ such that $|f(a)| < \min_{z\in\partial K}|f(z)|$. Then f has a zero in K.

Theorem 6.3 (The open mapping theorem)**.** *A non-constant holomorphic function* $f\colon U \to \mathbb{C}$ *is* open*, i.e.,* $f(V)$ *is open in* $\mathbb{C}$ *for any open set* $V \subset U$*.*

Warning 6.4. This statement is not to be mixed with the assertions

$$f^{-1}(\text{open}) \text{ is open} \quad \text{and} \quad f^{-1}(\text{closed}) \text{ is closed}$$

equivalent to the continuity of f. Note also that we do *not* assert that $f(K)$ is closed if $K \subset U$ is closed (nor is it true in general). However, if K is closed *in* $\mathbb{C}$ and *bounded*, then it is compact and $f(K)$ is also compact, hence closed. But this is a completely different story!

6.6 Schwartz' lemma

This is another specific application of the maximum modulus principle. We denote by $\mathbb{D} := \{|z| < 1\}$ the open unit disk.

Theorem 6.5 (Schwarz' lemma)**.** *Let* $f\colon \mathbb{D} \to \mathbb{D}$ *be a holomorphic function* (i.e., $|f(z)| < 1$ *for all* $z \in \mathbb{D}$)*, and assume that* $f(0) = 0$*. Then*:

$$|f'(0)| \leqslant 1, \qquad |f(z)| \leqslant |z| \quad \textit{for all } z \in \mathbb{D}.$$

Furthermore, if at least one inequality turns into equality (*for at least one* $z \in \mathbb{D}$)*, then* $f(z) = az$ *for some constant* $a \in \mathbb{C}$*,* $|a| = 1$*.*

This is often used in conjunction with the fractional linear transformations

$$g_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}, \quad \alpha \in \mathbb{D}, \tag{6.6}$$

which map $\mathbb{D}$ onto $\mathbb{D}$ biholomorphically (see §10.3).

7 Residues and integrals

7.1 Isolated singular points

A point $c \in \mathbb{C}$ is called an *isolated singular point* of a function f if f is defined and holomorphic in some punctured neighborhood $B_\varepsilon(c) \smallsetminus c$, $\varepsilon > 0$, of c. Due to

§5.1, the function f admits a Laurent expansion at c converging (at least) in the punctured disk $0 < |z - c| < \varepsilon$; it is this expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n \tag{7.1}$$

that we refer to when speaking about the *Laurent expansion at an isolated singular point* c (*cf.* Example 5.3).

The taxonomy of isolated singular points is as follows:

- *removable singularity*: f is bounded in a neighborhood of $c \iff a_n = 0$ for $n < 0$ in (7.1) $\iff f$ extends holomorphically through c;
- *pole of order* $m \geqslant 1$: $f \to \infty$ as $z \to c \iff a_{-m} \neq 0$ and $a_n = 0$ for $n < -m$ in (7.1) $\iff f(z) = (z - c)^{-m} g(z)$ for some g holomorphic in a neighborhood of c and such that $g(c) \neq 0$ (comparing to §6.2, poles are often regarded as zeroes of negative order; *cf.* also §7.4);
- *essential singularity*: for each $v \in \mathbb{C} \cup \infty$, there is a sequence $z_n \to c$ such that $f(z_n) \to v \iff$ in (7.1), there are infinitely many coefficients $a_n \neq 0$ with $n < 0$. A typical example is $c = 0$ for $e^{1/z}$.

A function holomorphic on a domain U except a number of isolated singular points *which are all removable or poles* is called *meromorphic*.

Remark 7.2. Not all singular points are isolated. For example, 0 for $1/\sin(1/z)$: singular points accumulate at 0. Even worse, 0 for $\ln z$ or $\sqrt{z}$: the function cannot even be defined in a whole punctured neighborhood of 0.

The *residue* of f at an isolated singular point c is defined as

$$\operatorname{res}_c f := a_{-1} \quad \text{in (7.1)}.$$

At a removable singularity, the residue is 0. At a pole c or order m,

$$\operatorname{res}_c f = \frac{1}{(m-1)!} \lim_{z \to c} \frac{d^{m-1}}{dz^{m-1}} \big[(z - a)^m f(z)\big].$$

Often, $(z - c)^m$ "cancels" with the "denominator" of f and limit reduces to mere substitution. (Accidentally, the order of a pole c is the order of vanishing at c of the denominator of f, whenever this makes sense.) Still, in most cases, you would not want to use this formula for $m > 1$, trying other tricks instead. (Many of such tricks are explored in the homework.) If $m = 1$, the formula simplifies to

$$\operatorname{res}_c f = \lim_{z\to c}(z-c)f(z).$$

There is no easy way to compute the residue at an essential singular point.

Warning 7.3. Although we speak about the residue of a function $f(z)$, it should be understood as the residue of the *holomorphic differential*:

$$\operatorname{res}_c f = \operatorname{res}_c f(z)dz.$$

This understanding is crucial if you make a change of variables more advanced then a mere shift $z \mapsto z - a$: the substitution is to be performed both in f and dz. In particular, this observation explains the slight deviation in the definition of the residue at infinity, see §7.2.

Remark 7.4. Sometimes, the easiest way to compute the residue is finding a few first terms of the Laurent expansion, manipulating (addition, subtraction, multiplication, and even division) with the Taylor series known from calculus.

7.2 Singular point at infinity

We say that ∞ is an isolated singular point of f if f is holomorphic in a *neighborhood of* ∞, *i.e.*, outer part $|z| > r$ of a sufficiently large disk. A Laurent expansion of f at ∞ is any expansion (7.1) converging on an unbounded annulus $|z-c| > r$. (Usually, one takes $c = 0$, but this is not compulsory.) The taxonomy is the same as in §7.1, except that, in terms of the Laurent expansion, one speaks about positive rather than negative terms. (*E.g.*, at a pole of order m, one has $a_m \neq 0$ and $a_n = 0$ for $n > m$.) In other words, one considers $g(w) := f(1/w)$ about its singular point 0. *However, the residue at infinity is defined differently*:

$$\operatorname{res}_\infty f := -a_{-1} \quad \text{in (7.1)},$$

cf. Warning 7.3. Note also that $\operatorname{res}_\infty f$ does *not* need to vanish if ∞ is a removable singularity (*i.e.*, $f(z)$ is bounded for $|z| \gg 0$). One can develop formulas similar (but not quite the same) to those in §7.1, but in most cases the following simple observations suffice:

- if $f(z) = O(z^{-2})$ as $z \to \infty$, then $\operatorname{res}_\infty f = 0$;

- if $f(z) = O(z^{-1})$ as $z \to \infty$, then $\operatorname{res}_\infty f = -\lim_{z\to\infty} zf(z)$.

Accidentally, I do not recommend trying to memorize all these formulas. Just imagine that most negative (respectively, positive) terms in (7.1) vanish and see what you can do (multiplying by powers of $(z-c)$, differentiating, *etc.*) to reach the coefficient a_{-1}.

7.3 The residue theorem

In this and next sections, we assume that $R \subset \mathbb{C}$ is a region (not necessarily connected or simply connected) with piecewise smooth boundary ∂R. The boundary is always assumed oriented so that R is "to the left"; roughly, the "outer" parts of the boundary are oriented counterclockwise, whereas its "inner" parts are oriented clockwise. The principal application of residues is the following theorem.

Theorem 7.5. *Assume that f is holomorphic in a neighborhood of a region R except for finitely many* (*necessarily isolated*) *singular points,* none of which is in *∂R. Then*

$$\int_{\partial R} f(z)dz = 2\pi i \sum_{c\in R} \operatorname{res}_c f.$$

Note that the summation in the right hand side is, in fact, finite, extending to the singular points of f only, as residues vanish at nonsingular points of f.

Theorem 7.5 still works of the region R is unbounded, provided that it contains a neighborhood of ∞. Certainly, in this case the residue $\operatorname{res}_\infty f$ should also be counted in the right hand side. In the extreme case $R = \mathbb{C} \cup \infty$ (and $\partial R = \varnothing$) we obtain the following useful statement.

Theorem 7.6. *Assume that f is holomorphic on the whole* extended *plane $\mathbb{C} \cup \infty$ except for finitely many* (*necessarily isolated*) *singular points. Then*

$$\sum_{c\in\mathbb{C}\cup\infty} \operatorname{res}_c f = 0.$$

Often, all *but one* residues of f are easily computed as explained in §7.1 (say, simple poles) or §7.2. Then, the remaining "bad" residue (*e.g.*, a multiple pole or essential singularity) can be found by Theorem 7.6.

Example 7.7. Both functions f in Example 4.2 have exactly two singular points, *viz.* $\pm i$. Hence,

$$\int_C f(z)dz = 2\pi i\big(\operatorname{res}_i f + \operatorname{res}_{-i} f\big) = -2\pi i \operatorname{res}_\infty f.$$

For the former function, $f(z) = O(z^{-2})$ as $z \to \infty$; hence, $\operatorname{res}_\infty f = 0$, see §7.2. For the latter, $f(z) = O(z^{-1})$ and $\operatorname{res}_\infty f = -\lim_{z\to\infty} zf(z) = -1$.

7.4 Logarithmic residues and the argument principle

If c is a zero or pole of f, so that $f(z) = (z-c)^m g(z)$ with $g(c) \neq 0$, then the *logarithmic residue*

$$\operatorname{res}_c \frac{f'}{f} = m =: \operatorname{ord}_c f$$

is the *order* of f at c. The next theorem is a consequence of Theorem 7.5.

Theorem 7.8 (the argument principle)**.** *Assume that f is meromorphic in a neighborhood of region R as in §7.3 and has neither zeroes nor poles on ∂R. Then*

$$\sum_{c\in R} \operatorname{ord}_c f = \frac{1}{2\pi i}\int_{\partial R} \frac{f'(z)dz}{f(z)}.$$

Here, the left hand side is the number of zeroes and poles of f in R, counted with multiplicities. The right hand side can vaguely be interpreted as

$$\frac{1}{2\pi i}\int_{\partial R} d(\ln f) = \frac{\ln f(\text{end}) - \ln f(\text{start})}{2\pi i} = \frac{\arg f(\text{end}) - \arg f(\text{start})}{2\pi}.$$

In other words, it is the number of full turns that the vector $f(z) \in \mathbb{C} \smallsetminus 0$ makes when z runs once along the boundary ∂R. This explains the name. Besides, since the increment of $\arg f(z)$ is an integer multiple of 2π, it can often be computed *approximately*, with $|\text{error}| < \pi$, using geometric arguments.

Below are a few applications of Theorem 7.8.

Theorem 7.9. *Assume that all functions $f_n \colon U \to \mathbb{C}$ are holomorphic and that* $f_n \to f \not\equiv 0$ uniformly on compacta. *Then any zero of f is a limit of zeroes of f_n. More precisely, if $f(c) = 0$, $c \in U$, then there is a sequence $c_n \to c$, $n \gg 0$, such that $f_n(c_n) = 0$.*

Theorem 7.10 (Rouché)**.** *Let f be as in Theorem 7.8, let g be meromorphic in a neighborhood of R, and assume that $|g(z)| < |f(z)|$ for all $z \in \partial R$. Then the number of zeroes and poles in R of $f + g$ equals that of f.*

Note that, in Theorem 7.10, we compare f and g *only on the boundary* ∂R. For example, taking for f the leading term $a_n z^n$ of a degree n polynomial $p(z)$, for g, the sum of the other terms, and for R, a sufficiently large disk, we establish once again the fundamental theorem of algebra: p has exactly n roots (same as $a_n z^n$), counted with multiplicities.

8 Application to real integrals

Since this section mainly deals with examples, for simplicity we assume that P is a real rational function of its arguments. Other assumptions are listed on the case-by-case basis, with no attempt to handle the "most general" situation.

8.1 Trigonometric functions over a period

This is a most straightforward application. Consider an integral of the form

$$I := \int_0^{2\pi} P(\sin x, \cos x)dx.$$

(We can take any other limits a, b with $b - a = 2\pi$.) Letting $z = e^{ix}$, we have

$$\sin x = \frac{z^2 - 1}{2iz}, \quad \cos x = \frac{z^2 + 1}{2z}, \quad dx = \frac{dz}{iz},$$

so that

$$I = \oint_{|z|=1} P\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z}\right)\frac{dz}{iz}.$$

This contour integral is usually easily computed by means of Theorem 7.5.

8.2 Rational functions

Consider an integral of the form

$$I := \int_{-\infty}^{\infty} P(x)dx,$$

and assume that $P(z) = O(z^{-2})$ as $z \to \infty$ and P has no real poles (so that the integral converges absolutely by the comparison tests from Calculus 101). We use Theorem 7.5 to compute the contour integral

$$\int_{-r}^{r} P(z)dz + \int_{C_r} f(z)dz,$$

where

$$C_r\colon z = Re^{it}, \quad t \in [0, \pi], \quad r = \text{const} \gg 0, \tag{8.1}$$

is a large semicircle. By the ML-bound, $\int_{C_r} \to 0$ as $r \to \infty$; hence, passing to the limit, we obtain

$$I = 2\pi i \sum_{\operatorname{Im} a > 0} \operatorname{res}_a P. \tag{8.2}$$

Remark 8.3. Like most other formulas, this one is not worth memorizing, even though it *looks* like Theorem 7.5 applied to the region $R := \{\operatorname{Im} z > 0\}$ whose boundary is $\mathbb{R}$. You should understand the approach! Besides, in some cases, one can use various symmetries of P and integrate over smaller regions in order to reduce the computation of residues. For example, if $P(z)$ has the form $Q(z^n)$, it may make sense to take for R the region bounded by the rays $0 \leqslant \arg z \leqslant e^{2\pi i/n}$ and appropriate arc of C_r.

8.3 Some transcendental functions

Here, we consider integrals that can be *reduced* (*via* Re or Im) to

$$I := \int_{-\infty}^{\infty} e^{\lambda i x} P(x)dx, \qquad \lambda \in \mathbb{R}, \quad \lambda > 0,$$

where $P(z) = O(z^{-1})$ as $z \to \infty$. Examples are $\sin x = \operatorname{Im} e^{ix}$ or $\cos x = \operatorname{Re} e^{ix}$; more complicated trigonometric expressions can be reduced to one or several of the above by means of trigonometric identities (most notable, power reduction). I strongly recommend mastering the examples:

$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x(1+x^2)}, \quad \int_{-\infty}^{\infty} \frac{\sin^2 x\, dx}{x^2}, \quad \int_{-\infty}^{\infty} \frac{\sin^3 x\, dx}{x^3}, \quad \int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} dx.$$

(For the last one, "at ∞" we combine this section and §8.4.)

We always assume that the *original real integrant* has no real poles; together with the bound $P(x) = O(x^{-1})$ this guarantees the absolute convergence of the integral. If also P has no real poles, we proceed as in §8.2, arriving at an analogue of (8.2). Instead of the ML-bound, the following lemma is used.

Lemma 8.4 (Jordan). *If $P(z) = O(z^{-1})$ as $z \to \infty$ and $\lambda > 0$, then* (*see* (8.1))

$$\lim_{r\to\infty} \int_{C_r} e^{\lambda iz} P(z)dz = 0.$$

If P does have real poles (as in all four examples above), we cannot integrate $e^{\lambda ix}P(x)$ along $[-r, r]$, as the integral would diverge. In this case, we circumvent each real pole a of P by a "small" *negatively oriented* semicircle

$$S_\varepsilon(a)\colon\ z = a + \varepsilon e^{it}, \quad t \in [\pi, 0], \quad 0 < \varepsilon = \text{const} \ll 1, \tag{8.5}$$

and analyze all limits $\lim_{\varepsilon\to 0} \int_{S_\varepsilon(a)}$ *in the hope that the part* Re *or* Im *that we are interested in does have a finite limit*. (Note that the full limit may be infinite!) This can be done, for example, using the partial (negative powers only) Laurent expansion at a, using the obvious observation that

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon(a)} (\text{a function analytic at } a)dz = 0$$

and computing the integrals of negative powers of z directly:

$$\int_{S_\varepsilon(a)} \frac{dz}{(z-a)^n} = \begin{cases} -\pi i, & \text{if } n = 1, \\ (1-n)^{-1}\big[\varepsilon^{1-n} - (-\varepsilon)^{1-n}\big], & \text{if } n > 1. \end{cases}$$

(For $n > 1$, we can use the anti-derivative.)

Example 8.6. The last of the four integrals above can be written as

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{ix - e^{ix}}{x^3} dx = \dots$$
$$= -\lim_{\varepsilon\to 0} \operatorname{Im} \int_{S_\varepsilon(0)} \frac{iz - e^{iz}}{z^3} dz = -\lim_{\varepsilon\to 0} \operatorname{Im} \int_{S_\varepsilon(0)} \left(-\frac{1}{z^3} - \frac{i^2}{2z} - \dots\right) dz = \frac{\pi}{2}.$$

Note that we have changed x to ix in order to be able to apply Im to the whole integral! (Separately, the two integrals would diverge at 0.) Note also that, since the integrant has no poles in the upper half plane, the whole contribution to the integral is from $S_\alpha(0)$. (This is the case in all above integrals except the first one.)

8.4 Key-hole shaped contours

This is a delicate subject that brings up the concept of multivalued functions and/or Riemann surfaces. As an example, consider an integral of the form

$$I := \int_0^\infty x^\alpha P(x)dx, \qquad \alpha \in \mathbb{R}, \quad -1 < \alpha < 0,$$

where $P(z) = O(z^{-1})$ as $z \to \infty$ and P has no poles in $[0, \infty)$. We want to integrate the function $f(z) := z^\alpha P(z)$, which is not quite well defined (and it is this fact that makes the integral computable!) Thus, we cut off the positive real axis, considering $\mathbb{C} \setminus [0, \infty)$, or else restricting the argument *via* $0 < \arg z < 2\pi$. However, since we do want f at real points, we extend slightly each edge of the cut making them overlap *but pretending that the overlapping parts do not intersect*! In other words, we restrict $-\delta < \arg z < 2\pi + \delta$ (and still $z \neq 0$), obtaining two *distinct* copies of the real semiaxis $(0, \infty)$, one with $\arg z = 0$, and one with $\arg z = 2\pi$, in which we allow f to take distinct values! (The result is a simplest example of the so-called *Riemann surface*, *i.e.*, something that locally looks like an open set in $\mathbb{C}$ but globally is not part of $\mathbb{C}$. Another, better known example is the Riemann sphere $\mathbb{C} \cup \infty$.)

In this new surface, we can integrate f along the "key-hole shaped" contour C consisting of a large circle

$$C_R \colon z = re^{it}, \quad t \in [0, 2\pi], \quad r = \text{const} \gg 0,$$

real line segment $[r, \varepsilon]$, small circle

$$S_\varepsilon \colon z = \varepsilon e^{it}, \quad t \in [2\pi, 0], \quad 0 < \varepsilon = \text{const} \ll 1,$$

and another segment $[\varepsilon, r]$. Note that the circles are no longer "closed" and the two segments are distinct, so that the integrals do not cancel. In fact, because of the argument difference (and the orientation), we have

$$\int_{[\varepsilon,r]} = \int_\varepsilon^r x^\alpha P(x)dx, \qquad \int_{[r,\varepsilon]} = -e^{2\pi i\alpha} \int_\varepsilon^r x^\alpha P(x)dx$$

On the other hand, in our Riemann surface, C bounds the open annular sector

$$R := \big\{\varepsilon < |z| < r,\ 0 < \arg z < 2\pi\big\},$$

to which we can apply Theorem 7.5. (A formal justification would consist in stepping a bit away from the border, applying the theorem, and taking limit.)

Since also $\int_{S_\varepsilon} \to 0$ as $\varepsilon \to 0$ and $\int_{C_r} \to 0$ as $r \to \infty$, both by the ML-bound, we arrive at

$$I = -\frac{\pi e^{\pi i \alpha}}{\sin \pi\alpha} \sum_{c \in \mathbb{C} \smallsetminus 0} \operatorname{res}_c f,$$

where the residues are computed under the assumption $0 < \arg z < 2\pi$. (Recall that P and, hence, f have no poles in $[0, \infty)$. The coefficient in front of the sum is the "classical" way to write the quotient $2\pi i/(1 - e^{2\pi i \alpha})$.)

Remark 8.7. Instead of working with the overlapping contour, we could have stepped away from the boundary (say, considering the shifted segments $[\varepsilon, r] \pm \delta i$) and passed to yet another limit $\delta \to 0$ at the end. However, this approach would be much more technical and somewhat less clean.

Remark 8.8. A similar approach applies to "indefinite" integrals, *i.e.*, those of the form $\int_a^\infty P(x)dx$: we integrate the function $f(z) := \ln(z - a)P(z)$. Try to work out the details.

Remark 8.9. For functions involving

$$\left(\frac{x-a}{b-x}\right)^\alpha \quad \text{or} \quad \ln\frac{x-a}{b-x},$$

one can try a similar contour consisting of $[a + \epsilon, b - \epsilon]$, $[b - \epsilon, a + \epsilon]$, and two small circles of radius ϵ about a and b.

8.5 Other contours

There is a huge number of other tricks, all consisting in applying Theorem 7.5 to an appropriate contour C and passing to appropriate limits, so that integrals over some parts of C tend to zero, whereas others tend to the integral in question.

Example 8.10. The integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x} dx}{e^x + 1}, \quad 0 < \alpha < 1,$$

can be computed *via* the integration over the (sides of the) rectangle with the vertices $\pm r$, $\pm r + 2\pi i$ and letting $r \to \infty$. Fill in the details.

8.6 Summation of series

The following trick can be used to sum up some numeric series. We observe that the function $\pi \cot \pi z$ has simple poles at all integers, with all residues equal to 1.

Lemma 8.11. *If $P(z) = O(z^{-2})$ as $z \to \infty$, then*

$$\int_{\square_N} P(z) \cot \pi z \, dz \underset{N\to\infty}{\longrightarrow} 0,$$

where $\square_N$ is the square with the vertices $(N + \frac{1}{2})(\pm 1 \pm i)$, $N \in \mathbb{N}$.

Let S be the set of all poles of P. Then, applying Theorem 7.5 to $\square_N$ and letting $N \to \infty$, we obtain the formula

$$\sum_{n \in \mathbb{Z} \setminus S} P(n) = -\pi \sum_{c \in S} \operatorname{res}_c\big(P(z) \cot \pi z\big). \tag{8.12}$$

Example 8.13. The most "classical" application of (8.12) is the computation of the ζ-*values*: letting $P(z) := z^{-2k}$, $k \in \mathbb{Z}$, $k \geqslant 0$, and using the symmetry, we obtain

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2} a_{2k-1} \pi^{2k},$$

where $\sum_{n=-1}^{\infty} a_n z^n$ is the Laurent series of $\cot z$ at 0. Given time, any number of coefficients can be computed, *e.g.*, dividing the series for $\cos z$ and $\sin z$. In a sense, these coefficients are "known":

$$a_{2k-1} = (-1)^k \frac{2^{2k} B_{2k}}{(2k)!},$$

where B_n are the so-called Bernoulli numbers. (All coefficients except a_{-1} are negative.) Note that the odd ζ-values $\zeta(2k+1)$ are not known!

9 The Γ-function

The Γ-function is defined *via*

$$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} dx.$$

This integral converges (and thus defines Γ) for $\operatorname{Re} z > 0$. It can be shown that Γ is holomorphic, with the expected

$$\Gamma'(z) := \int_0^\infty x^{z-1} \ln x \, e^{-x} dx.$$

(Since the integral is improper, this needs some work, but not too much.)

Integrating by parts, we obtain the *functional equation*

$$z\Gamma(z) = \Gamma(z+1). \tag{9.1}$$

Since obviously $\Gamma(1) = 1$, this equation implies that $\Gamma(n) = (n-1)!$ for each positive integer n, *i.e.*, Γ is a "natural" extension of the factorial function to complex arguments. Another known value of Γ is $\Gamma(1/2) = \sqrt{\pi}$.

Using (9.1), one can also extend Γ to $\operatorname{Re} z \leqslant 0$, by letting

$$\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)},$$

where $n \geqslant 0$ is an integer such that $\operatorname{Re}(z+n) > 0$. (In view of (9.1), this value does not depend on the choice of such n.) The result is a meromorphic function that has a simple pole at each *integer* $n \leqslant 0$.

The B-function (this is "Beta", not B) is defined *via*

$$\mathrm{B}(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1} dx;$$

this integral converges for $\operatorname{Re} p > 0$ and $\operatorname{Re} q > 0$. The most famous property of this function is its relation to Γ:

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

The Γ-function, either *per se* or *via* B, can be used to compute a great deal of definite integrals, usually by an appropriate change of variables, which may be followed by a contour argument similar to §8.

Example 9.2. Substituting $\sin^2 t = x$, one has

$$\int_0^{\pi/2} \sin^{2p-1} t \cos^{2q-1} t \, dt = \mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

10 Automorphisms of simply connected domains

According to *Riemann's mapping theorem*, there are essentially three distinct simply connected domains: the Riemann sphere $\mathbb{P}^1 := \mathbb{C} \cup \infty$, the plane $\mathbb{C}$, and the unit disk $\mathbb{D} := \{|z| < 1\}$, which is conformally equivalent to any other simply connected *proper* open subset $U \subsetneq \mathbb{C}$.

We are interested in the *conformal transformations* (or *biholomorphisms*) of these domains U, *i.e.*, bijective holomorphic maps $f\colon U \to U$. (The injectivity implies that $f'(z) \neq 0$ for all $z \in U$ and, hence, f^{-1} is also holomorphic.) Such transformations preserve angles between curves but, in general, distort length.

10.1 The Riemann sphere

It is convenient to follow the tradition and denote $\mathbb{P}^1 := \mathbb{C} \cup \infty$, so that ∞ is no longer distinguished. Any conformal transformation of $\mathbb{P}^1$ is fractional linear:

$$f_A\colon z \mapsto \frac{az+b}{cz+d}, \qquad A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{C}). \tag{10.1}$$

(It is convenient to parametrize these maps by (2×2)-matrices rather than by quadruples of complex numbers.) Clearly, $f_A \neq \mathrm{const}$ if and only if $\det A \neq 0$ and proportional matrices A and αA, $\alpha \in \mathbb{C} \smallsetminus 0$, define the same map $f_A = f_{\alpha A}$. Furthermore, the composition of two fractional linear transformations is also of the same form:

$$f_A \circ f_B = f_{AB}.$$

It follows that the group of conformal automorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ is the so-called *general projective group* *PGL*$(2, \mathbb{C})$, *i.e.*, the group of nonsingular ($\det A \neq 0$) complex (2×2)-matrices A modulo the scalar matrices αI, $\alpha \in \mathbb{C} \smallsetminus 0$.

The action of *PGL*$(2, \mathbb{C})$ on $\mathbb{P}^1$ is 3-*transitive*, *i.e.*, any triple of pairwise distinct points can be taken to any other such triple. In fact, the map

$$z \mapsto \frac{(z_3 - z_1)(z - z_2)}{(z_3 - z_2)(z - z_1)} \tag{10.2}$$

takes (z_1, z_2, z_3) to $(\infty, 0, 1)$, and this is the *only* map (10.1) with this property. A fourth point z_4 is taken by (10.2) to

$$\lambda = (z_1, z_2; z_3, z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)};$$

this expression is called the *cross-ratio* of the quadruple z_1, z_2, z_3, z_4. The cross-ratio is preserved by fractional linear transformations, and it is the only invariant that distinguishes ordered quadruples. See cross-ratio in Wikipedia.

Fractional linear transformations take lines or circles to lines or circles. (Note that a line can be taken to a circle and *vice versa*; in fact, it is convenient to treat lines as "circles" through ∞.) Any triple z_1, z_2, z_3 of pairwise distinct points lies in a unique line or circle L, which divides $\mathbb{P}^1$ into two connected regions. A fourth point z_4 lies in L if and only if $\lambda := (z_1, z_2; z_3, z_4) \in \mathbb{R}$, *i.e.*, $\operatorname{Im}\lambda = 0$; otherwise, the two components of $\mathbb{P}^1 \smallsetminus L$ are distinguished by the sign of $\operatorname{Im}\lambda$.

10.2 The complex plane

This domain is boring. Any conformal transformation of $\mathbb{C}$ is affine linear:

$$z \mapsto az + b, \qquad a, b \in \mathbb{C},\ a \neq 0.$$

Thus, *a posteriori*, conformal transformations of $\mathbb{C}$ are those of $\mathbb{P}^1$ that fix the distinguished point ∞. The action is 2-transitive; lines are taken to lines, and circles are taken to circles. Geometrically, any transformation is a composition of translation, rotation, and dilation.

10.3 The unit disk

Any conformal transformation $\mathbb{D} \to \mathbb{D}$ has the form

$$z \mapsto \beta \frac{z - \alpha}{1 - \bar{\alpha} z}, \qquad \alpha \in \mathbb{D},\ \beta \in \partial\mathbb{D}, \tag{10.3}$$

i.e., it is the composition of a certain g_α as in (6.6) and rotation $z \mapsto \beta z$; these rotations are all maps (10.3) preserving the origin. Again, *a posteriori* we conclude that any transformation extends to $\mathbb{P}^1$ and is characterized by the property that it preserves the boundary circle $\partial\mathbb{D}$ (as a set) and takes its interior to itself.

The action is $1\frac{1}{2}$-transitive: any *flag* $(\text{point}, \text{direction})$ can be taken to any other flag. Lines or circles are taken to lines or circles, preserving the angle with the boundary circle $\partial\mathbb{D}$.

Often, it is more convenient to consider an alternative model, *viz.* the upper half-plane $\mathbb{H} := \{\operatorname{Im} z > 0\}$, which is taken to $\mathbb{D}$, *e.g.*, by the *Cayley transform*

$$z \mapsto \frac{z - i}{z + i}. \tag{10.4}$$

The conformal transformations $\mathbb{H} \to \mathbb{H}$ are those maps f_A in (10.1) that preserve the boundary $\partial\mathbb{H} = \mathbb{R}$ and the sign of $\operatorname{Im} z$. It is easy to see that, after rescaling, one must have $a, b, c, d \in \mathbb{R}$ and $\det A > 0$. Rescaling further to $\det A = 1$, we arrive at the so-called *special projective group* $\mathit{PSL}(2, \mathbb{R})$, *i.e.*, the group of unimodular ($\det A = 1$) real (2×2)-matrices A modulo $\pm I$.

10.4 Hyperbolic geometry

Redefine the length of a curve $C \subset \mathbb{H}$ and distance between $z, w \in \mathbb{H}$ *via*

$$\ell(C) := \int_C \frac{ds}{\operatorname{Im} z}, \qquad \operatorname{dist}(z, w) := \inf \ell(C),$$

where the infimum is taken over all curves $C \subset H$ connecting z and w. The result is called the *Poincaré metric*, and the pair $(\mathbb{H}, \operatorname{dist})$ is referred to as the *Poincaré half-plane model* of hyperbolic geometry. (Using Cayley transform (10.4), we can transfer this metric to the unit disc $\mathbb{D}$, obtaining the *Poincaré disk model*.)

With respect to this metric, *lines* (*aka geodesics*, *i.e.*, shortest curves) in $\mathbb{H}$ are the usual Euclidean lines and half-circles orthogonal to the *absolute* $\mathbb{R} = \partial\mathbb{H}$. These lines satisfy all Euclidean axioms (*e.g.*, two lines intersect in at most one point, through two distinct points there is a unique line, *etc.*) except the famous parallel postulate: now, *for any line L and any point $z \notin L$, there are* at least two *lines through z disjoint from L.*

It turns out that the group $\mathit{PSL}(2, \mathbb{R})$ of conformal transformations of $\mathbb{H}$, see §10.3, is precisely the group of orientation preserving isometries (rigid motions) of the hyperbolic plane $(\mathbb{H}, \operatorname{dist})$. The isometries $\operatorname{id} \neq f_A$, $A \in \mathit{PSL}(2, \mathbb{R})$ can be classified according to the *trace* $\operatorname{trace} A := a + d$, or, equivalently, according to the number and position of their fixed points:

- if $|\operatorname{trace} A| < 2$, then f_A has a unique fixed point $a \in \mathbb{H}$ (and the conjugate point $\bar{a} \in \bar{\mathbb{H}}$); this is an analogue of Euclidean rotation about a;
- if $|\operatorname{trace} A| > 2$, then f_A has two distinct *ideal* fixed points $a, b \in \partial\mathbb{H}$; this is an analogue of Euclidean translation along the line through a and b;
- if $|\operatorname{trace} A| = 2$, then f_A has a single 2-fold ideal fixed point $a \in \partial\mathbb{H}$; this *horolation* (“rotation” about a) has no analogues in the Euclidean plane.

Technically, in the last two cases f_A has no fixed points at all, as the absolute $\partial\mathbb{H}$ is *not* part of the hyperbolic plane: the ideal points $a \in \partial\mathbb{H}$ correspond to pencils of parallel lines in $\mathbb{H}$. (Note, though, that the concept of parallel in hyperbolic geometry differs from that in Euclidean.) However, these ideal points are very useful in the description of many geometric phenomena.

For details and references, see, *e.g.*, Poincaré half-plane model in Wikipedia.