

MATH 132 Lecture notes

BINARY RELATIONS

1. (Binary) relations

Informally, a *(binary) relation* from a set A to a set B is a “property” R with two arguments $a \in A$ and $b \in B$. In other words, for each pair $a \in A$ and $b \in B$, we know whether $R(a, b)$ is true or false. Here, the word “binary” signifies that fact that the relation has **two** arguments. One can also consider *unary*, *ternary*, ... *n-ary* relations, but we typically stick to the binary ones and omit the prefix.

It is common to use the *infix notation* for **binary** relations:

$$\begin{aligned} R(a, b) \text{ is true:} & \quad a R b, \\ R(a, b) \text{ is false:} & \quad a \not R b. \quad (\text{not used very often}) \end{aligned}$$

Here are a few examples justifying this notation (and explaining that, in fact, we are well familiar with binary relations):

$a \leq b$	$A = B = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \text{ etc.}$	R AT
$a = b$	$A = B$ is any set,	RS AT
$a \equiv b \pmod{m}$	$A = B = \mathbb{Z}$ and $m \in \mathbb{Z}^+$ fixed ,	RS T
$a \mid b$	$A = B = \mathbb{Z}^+$	R AT

or, more down to earth

a is a parent of b	$A = B = \{\text{people}\},$	A
a takes b	$A = \{\text{students}\}, B = \{\text{courses}\},$	
b is taken by a	same sets,	
a is in b	$A = \{\text{cities}\}, B = \{\text{states}\},$	
a is the capital of b	same sets,	
b has capital a	same sets.	

In the first four examples, $A = B$ is the same set; in this case, we speak about a (binary) relation **on** A .

Another example of binary relations, for which we do **not** use the infix notation, is given by functions:

$$\text{if } f: A \rightarrow B, \quad \text{then } a f b \text{ is a synonym for } b = f(a).$$

Though, there **is** a common “infix” notation for functions: $a \xrightarrow{f} b$. For example, the relations “is in” and “has capital” above **are** functions, are they not? Are there any other functions above?

Thus, we can say that binary relations generalize functions; this fact explains the prepositions “**from**” and “**to**” in the name.

Our immediate goal is to study in detail certain **special classes** of relations, most notably

- functions (done before);
- *(partial) orders* (RAT), e.g., $\leq, \mid, \subseteq, \preceq, \triangleleft$;
- *equivalence relations* (RST), e.g., $=, \equiv, \sim, \cong, \approx, \simeq$.

2. Description and formal definition

The “easiest” way to describe a binary relation is by its *truth table*. For example, $\leq$ on $A = B = \{1, 2, 3\}$:

	1	2	3	B
1	✓	✓	✓	
2		✓	✓	
3			✓	
A				

Or, “is the capital of” and “is in”:

	CA	NV	NY	...	B
Albany			✓		
Carson City		✓			
Las Vegas		○			
LA	○				
NY			○		
Sacramento	✓				
San Francisco	○				
⋮					
A					

As yet another example, in **GAP** there are **three** Boolean values: **true**, **false**, and **fail**. There is no obvious “logical” way to compare them; thus, the order (*i.e.*, relation $\leq$) on the set $A = \{\text{true}, \text{false}, \text{fail}\}$ is **defined** by the table

	true	false	fail	B
true	✓	✓	✓	
false		✓	✓	
fail			✓	
A				

Now, we all understand what the **rigorous mathematical definition** of a relation from A to B should be. (As usual in math, we start with an intuitively clear notion and end up with a rigorous, but incomprehensible definition.) The cells of our tables are indexed by pairs (a, b) , *i.e.*, elements of the *Cartesian product* $A \times B$. Some of these cells (elements) are marked, meaning that a subset $R \subseteq A \times B$ is chosen.

Definition 2.1. A (*binary*) *relation* from a set A to a set B is a subset $R \subseteq A \times B$. We write

$$a R b \quad \text{iff} \quad (a, b) \in R.$$

The notation $a R b$ is a bit awkward. However, when we use more conventional symbols like $a \leq b$ or $a = b$, we also use **the same** symbol to denote the corresponding subset of $A \times B$, *i.e.*, we have to write

$$\leq \subseteq A \times B, \quad = \subseteq A \times B, \quad \text{etc.},$$

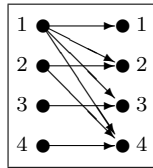
which is equally unusual.

As yet another explanation, a relation R is a function defined **on pairs** (a, b) and taking values in the two-element set $\{\mathbf{false}, \mathbf{true}\}$. We know that pairs (a, b) are elements of $A \times B$. On the other hand, we can identify $\{\mathbf{false}, \mathbf{true}\}$ with the two-element set $\{0, 1\}$, and we also know that functions $S \rightarrow \{0, 1\}$ are “the same” as subsets of S . Thus, a binary relation **is** indeed a subset of $A \times B$.

Example 2.2. From this point of view, a function $f: A \rightarrow B$ is **nothing but its graph**

$$f = \{(a, b) \in A \times B \mid b = f(a)\}.$$

Like functions, general relations can be represented by arrows, but we no longer require that there must be one and only one outgoing arrow from each point of the domain. For example, the relation $\mid$ (“divides”) on the set $A = \{1, 2, 3, 4\}$ can be represented like this:

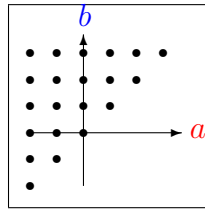


We draw an arrow from a to b whenever $a \mid b$.

This is the formal
definition of a
function: a subset
of $A \times B$ with
certain properties

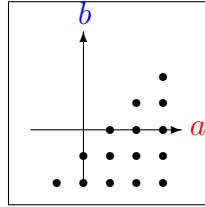
Here are more examples of relations on $A = \mathbb{Z}$ (I cannot do shades in $\text{\TeX}$, but you can imagine the pictures for “the same” relations on $\mathbb{R}$):

$$R_1: a \leq b$$



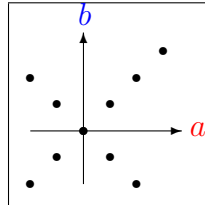
R AT

$$R_2: a > b$$



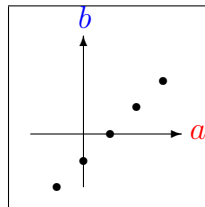
AT

$$R_3: a = b \text{ or } a = -b$$



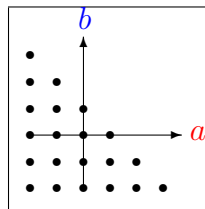
RS T

$$R_4: a = b + 1$$



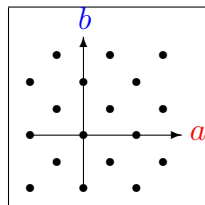
A

$$R_5: a + b \leq 1$$



S

$$R_6: a \equiv b \pmod{2}$$



RS T

3. Properties of relations

Here we discuss certain **extra** properties that a particular relation may or may not have. We speak about binary relations **on a set** A .

Definition 3.1. A relation R on A is *reflexive* if $a R a$ for any $a \in A$. In other words, R contains the *diagonal* $\{(a, a) \mid a \in A\}$.

Example 3.2. These relations are reflexive:

$$\begin{array}{ll} | & \text{on } \mathbb{Z}^+ \quad \text{we know that } a | a \text{ for any } a \in \mathbb{Z}^+, \\ \leq & \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R} \quad \text{pretty much by definition,} \\ = & \text{also the definition.} \end{array}$$

The relation $<$ is *not* reflexive: *e.g.*, it is not true that $1 < 1$.

As usual, to prove that a relation R is reflexive, you **MUST** consider an **arbitrary** element $a \in A$ (not just a few examples); to prove that it is **not**, a **single** counterexample is enough! (*E.g.*, in the last example, $a < a$ is *never* true, but we do not need that much!)

Definition 3.3. A relation R on A is *symmetric* if

$$a R b \text{ implies } b R a$$

for any pair $a, b \in A$. In other words, R is symmetric with respect to the involution $A \times A \rightarrow A \times A$, $(a, b) \mapsto (b, a)$. (In the pictures with the plane, this is the symmetry against the diagonal $a = b$.)

A relation is *anti-symmetric* if

$$(a R b \text{ and } b R a) \text{ imply } a = b.$$

Note that it is **not** true that any relation is either symmetric or anti-symmetric: a “typical” relation is neither!

Example 3.4. These relations are symmetric:

$$\begin{array}{l} a = b \quad \text{on any set,} \\ a = b \text{ or } a = -b \quad \text{on any set where this makes sense,} \\ a + b \leq 1 \quad \text{on any set where this makes sense,} \\ a \text{ has the same color as } b \quad \text{on the set of cars,} \\ |A| = |B| \quad \text{on the power set } \mathcal{P}(S). \end{array}$$

These are anti-symmetric:

$$\begin{array}{l} a = b \quad \text{on any set,} \\ a \leq b \quad \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \\ a | b \quad \text{on } \mathbb{Z}^+, \\ A \subseteq B \quad \text{on the power set } \mathcal{P}(S), \\ a = b + 1 \quad \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \\ a < b \quad \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}. \end{array}$$

(Concerning the last two, observe that, if $a R b$ and $b R a$ never happens simultaneously, *the relation is anti-symmetric by definition!*) Finally, this one is neither symmetric nor anti-symmetric:

$$a | b \quad \text{on } \mathbb{Z}, \quad 1 | 2, 2 \nmid 1; \text{ but } 1 | (-1) \text{ and } -1 | 1.$$

Note that $=$ is the only relation both symmetric and anti-symmetric!

Definition 3.5. A relation R on A is *transitive* if

$$(a R b \text{ and } b R c) \text{ imply } a R c.$$

for any triple $a, b, c \in A$. (Later we will see that this is equivalent to saying that $R^2 \subseteq R$.)

Example 3.6. These relations are transitive:

$$\begin{aligned} a &= b && \text{on any set,} \\ a &= b \text{ or } a = -b && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \quad (\text{needs proof!}) \\ a &\leq b && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \\ a &< b && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \\ a &| b && \text{on } \mathbb{Z}^+ \text{ or } \mathbb{Z}, \\ a &\equiv b \pmod{m} && \text{on } \mathbb{Z}, \text{ with } m \text{ fixed,} \\ a &\text{ has the same color as } b && \text{on the set of cars,} \\ |A| &= |B| && \text{on the power set } \mathcal{P}(S), \\ A &\subseteq B && \text{on the power set } \mathcal{P}(S). \end{aligned}$$

These are not:

$$\begin{aligned} a + b &\leq 1 && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \quad (a = 1, b = 0, c = 1) \\ a &= b^2 && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \quad (a = 16, b = 4, c = 2) \\ a &= b + 1 && \text{on } \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \quad (a = 2, b = 1, c = 0) \\ a &\text{ is a parent of } b && \text{on the set of people.} \end{aligned}$$

Again, to prove that R is **not** transitive, just **one** counterexample would suffice!

Remark 3.7. As already mentioned, we will mainly consider two special classes of relations:

- *partial orders* = reflexive, **anti**-symmetric, and transitive, or
- *equivalence relations* = reflexive, symmetric, and transitive.

As you will see, the change “symmetric” $\leftrightarrow$ “anti-symmetric” makes a whole world of difference!

4. Counting relations

Example 4.1. How many binary relations are there on an n -element set A ?

Solution: A relation is a subset of A^2 , i.e., an element of $\mathcal{P}(A^2)$. We have

$$|A^2| = n^2, \quad \text{hence} \quad |\mathcal{P}(A^2)| = 2^{n^2}.$$

As a reminder, the *superpower* 2^{n^2} should be understood as $2^{(n^2)}$, not as $(2^n)^2 = 2^{2n}$. Though, most reasonable computer languages would not accept 2^{n^2} without parentheses!

Example 4.2. How many **reflexive** relations are there on an n -element set A ?

Solution: We should be slightly more careful. We still have the n^2 “cells” in the “table” A^2 . However, this time, due to the reflexivity, the n diagonal cells (a, a) , $a \in A$, are “checked” automatically. Hence, we only have the freedom to choose the “check marks” for the remaining $n^2 - n$ cells, two choices for each, and the answer is $2^{n^2 - n}$.

Example 4.3. How many **anti-symmetric** relations are there on an n -element set A ?

Solution: We should be even more careful! Following the definition, we can choose whatever we want, i.e., **false/true**, for any of the n diagonal cells. The rest splits into $n(n-1)/2$ pairs of symmetric cells

$$(a, b), \quad (b, a), \quad a \neq b.$$

Normally, for each pair we would have four choices:

$$\{\text{false}, \text{true}\} \times \{\text{false}, \text{true}\}.$$

However, since $a \neq b$, the choice **(true, true)** is now forbidden and we are left with only three:

$$(\text{false}, \text{false}), \quad (\text{false}, \text{true}), \quad (\text{true}, \text{false}),$$

which are all allowed. Thus, by the product rule, we have

$$2^n \cdot 3^{n(n-1)/2}.$$

Along these lines, one can count the number of reflexive/irreflexive and/or symmetric/anti-symmetric relations on an n -element set. (**Try this at home!**) However, there is no known formula for the number of *transitive* relations! For small values of n , one can try to list them one by one, but the numbers are overwhelming:

$$T(4) = 3,994, \quad T(5) = 154,303, \quad T(6) = 9,415,189$$

(according to the Book). They are only known for $n \leq 17$.

5. Combining relations

Relations are **subsets** of $A \times B$ and, as such, they are subject to all standard subset operations. Here are the operations and their meaning “in plain English.” Let $R, S \subset A \times B$ be two relations. Then:

$$\begin{aligned}
 a(R \cup S)b & \text{ means } aRb \text{ or } aSb, \\
 a(R \cap S)b & \text{ means } aRb \text{ and } aSb, \\
 a(R - S)b & \text{ means } aRb \text{ but not } aSb, \\
 a(R \oplus S)b & \text{ means } aRb \text{ xor } aSb, \text{ either } R \text{ or } S \text{ but not both} \\
 a(\bar{R})b & \text{ means not } aRb, \text{ or } a \not R b.
 \end{aligned}$$

Here

$$R \oplus S := (R \cup S) - (R \cap S) \quad (\text{more often } R \triangle S)$$

is the so-called *symmetric difference*; it is used but not introduced in the Book.

Example 5.1. Let $R = \leq$ and $S = \geq$. Then:

$$\begin{aligned}
 a(R \cup S)b & \text{ means any pair } (a, b) \text{ is related,} \\
 a(R \cap S)b & \text{ means } a = b \text{ a common way to prove equalities} \\
 a(R - S)b & \text{ means } a < b, \\
 a(R \oplus S)b & \text{ means } a \neq b, \\
 a(\bar{R})b & \text{ means } a > b.
 \end{aligned}$$

Now, recall that relations generalize functions, and for functions we also have *composition*:

given $A \xrightarrow{f} B \xrightarrow{g} C$, we have $g \circ f: A \longrightarrow C$, $a \mapsto g(f(a))$.

For relations, we do not have a unique $f(a)$ (maybe even none at all); hence, we define composition slightly differently.

Definition 5.2. Given relations R from A to B and S from B to C , the *composition* is the relation $S \circ R$ from A to C defined *via*

$$a (S \circ R) c \text{ if and only if } \exists b \in B: (a R b) \ \& \ (b S c).$$

For functions, the b above is unique (determined by a) and we have the same old composition of functions. Note that, for S and R to be composable, in this order, the target of R and the source of S must be the same set B .

Example 5.3. Let

$$A = \{\text{people}\} \quad \text{and} \quad R = \text{“is a parent of” on } A.$$

Then

$$R^2 := R \circ R = \text{“is a grand-parent of” on } A.$$

Theorem 5.4. *As in the case of functions, composition of relations is associative: if*

$$R \text{ is from } A \text{ to } B, \quad S \text{ is from } B \text{ to } C, \quad T \text{ is from } C \text{ to } D,$$

then

$$(T \circ S) \circ R = T \circ (S \circ R).$$

Proof. Just spell out the definitions: both relations are

$$a (T \circ S \circ R) d \text{ iff } \exists b \in B, \exists c \in C: (a R b) \ \& \ (b S c) \ \& \ (c T d). \quad \square$$

For this reason, we omit parentheses in iterated compositions.

If R is a relation **on** a set A , we can define its *powers*:

$$R^1 = R, \quad R^n = R^{n-1} \circ R \quad \text{for } n \geq 2.$$

That is,

$$R^n = \underbrace{R \circ \dots \circ R}_{n \text{ factors}}.$$

By [Theorem 5.4](#), the iterated composition does not depend on the order of operations.

Remark 5.5. Probably, R^0 should be defined as the *identity relation* $=$, which is the diagonal $\{(a, a) \mid a \in A\}$. Note, accidentally, that

$$= \circ R = R = R \circ =$$

for any relation R .

Example 5.6. Let

$$A = \{\text{people}\} \quad \text{and} \quad R = \text{“is a parent of” on } A.$$

Then

$$\begin{aligned} R^2 &= \text{“is a grand-parent of” on } A, \\ R^3 &= \text{“is a great grand-parent of” on } A, \\ R^4 &= \text{“is a great great grand-parent of” on } A, \dots \end{aligned}$$

Try to see that

$$S := \bigcup_{n=1}^{\infty} R^n = \text{“is an ancestor of”}.$$

This is the
so-called *transitive
closure* of R

Theorem 5.7. A relation R on A is *transitive* if and only if

$$R^n \subseteq R \quad \text{for each } n \geq 2.$$

Proof. “If”: In fact, $n = 2$ is enough. Moreover, $R^2 \subseteq R$ is merely a **restatement** of the definition of transitivity (see [Definition 3.5](#)):

$$\underbrace{\exists b \in A: (a R b) \ \& \ (b R c)}_{a R^2 c} \implies \underbrace{a R c}_{a R c}.$$

Note also that $\Rightarrow$ for relations as “properties” means $\subseteq$ for relations as subsets of A^2 .

“Only if” is proved by induction. Assume that R is transitive.

Basic step (the case $n = 2$): as just shown, this is **equivalent** to the transitivity of R . ✓

Inductive step: assume that $R^k \subseteq R$. Then

$$R^{k+1} = R^k \circ R \stackrel{\text{IH}}{\subseteq} R \circ R = R^2 \stackrel{\text{BS}}{\subseteq} R. \quad \checkmark \quad \square$$

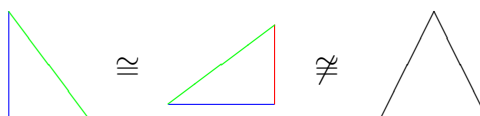
Example 5.8. The relation $S = \text{“is ancestor of”}$, see [Example 5.6](#), is transitive:

$$S^2 = \bigcup_{n=2}^{\infty} R^n \subset S.$$

Note, accidentally, that the inclusion is *proper*: $a S^2 b$ if and only if b is at least a grand-parent of a (the “first order” parents are missing). Thus, R^2 in [Theorem 5.7](#) does not need to be **equal** to R , even though quite often it is.

Thus, `myfile` and `MyFile` would refer to the same file, although the name (given when the file was created) would be retained (and shown in the listings) *including the case*. Here, we can **identify** the equivalence classes with UPPERCASE STRINGS. That is what older file systems did: the name was converted and stored in the upper case, so that there was no way to tell what name was given when the file was created.

Example 6.5. In high school geometry, two triangles are called *congruent* if one can be taken to the other by a rigid motion of the plane:



(We cannot say that the triangles are *equal*: a triangle is defined as a *set of points*, and, *as sets*, they are most likely distinct as they consist of distinct points!) This is an equivalence relation:

- $a = \text{id}(a)$, and id is a rigid motion;
- if $b = f(a)$, then $a = f^{-1}(b)$, and the inverse f^{-1} of a rigid motion f exists and is also a rigid motion;
- if $b = f(a)$ and $c = g(b)$, then $c = (g \circ f)(a)$, and the composition $g \circ f$ of two rigid motions f, g is again a rigid motion.

This is a common argument: the equivalence classes appear as the *orbits* of a *group action*

We can say that two triangles are *congruent* if they have the same “*shape*”. However, it is not really easy to explain what precisely the *shape* of a triangle is. Therefore, we **define** a *triangular shape* as a class of congruent triangles!

Another common thing: *define* a property by *listing* the objects that have it! Only mathematicians can do that!

Example 6.6. You may remember that we never **defined** the cardinality of a set: we only defined what it means to *have the same cardinality*:

$$|A| = |B| \quad \text{iff} \quad \exists \text{ bijection: } A \leftrightarrow B.$$

This,

$$A \sim B \quad \text{iff} \quad |A| = |B|, \text{ i.e., } \exists \text{ bijection: } A \leftrightarrow B,$$

is an equivalence relation. (I think we proved that, without mentioning the name.) To avoid logical difficulties, we should stick to the subsets of a certain *universe* $\mathcal{U}$: then, $\sim$ is a relation on $\mathcal{P}(\mathcal{U})$. Now, we can **define** a *cardinality* as a *class of unipotent sets*! Thus,

$$\begin{aligned} \aleph_0 &= \{\text{all subsets of } \mathcal{U} \text{ equipotent to } \mathbb{Z}^+\}, \\ c &= \{\text{all subsets of } \mathcal{U} \text{ equipotent to } \mathbb{R}\}, \dots \end{aligned}$$

Accidentally, the *natural numbers* themselves are formally **defined** as *cardinalities of finite sets* (where a set is called *finite* if it is not equipotent to its proper subset).

Well, this is **too** much of formal logic!

6.1. Equivalence classes. Now, let us treat equivalence classes more formally. Throughout this section, we *fix an equivalence relation $\sim$ on a set A* .

Definition 6.7. The *equivalence class* of an element $a \in A$ is the set of all elements $b \in A$ equivalent to a :

$$[a]_{\sim} = [a] := \{b \in A \mid a \sim b\}.$$

(We drop the subscript $\sim$ whenever the relation is understood.)

Plenty of examples have been given above. Note that **each equivalence class is nonempty**: in fact, by the **reflexivity** of $\sim$, we have

$$(6.8) \quad a \in [a] \quad \text{for any } a \in A.$$

Theorem 6.9. For $a, b \in A$, the following statements are equivalent:

- (1) $a \sim b$;
- (2) $[a] = [b]$;
- (3) $[a] \cap [b] \neq \emptyset$.

Proof. (1) $\implies$ (2). Assume that $a \sim b$. Then:

$$c \in [b] \xLeftrightarrow{\text{def}} b \sim c \xRightarrow{\text{T}} a \sim c \xLeftrightarrow{\text{def}} c \in [a], \quad \text{i.e.,} \quad [b] \subseteq [a].$$

By the **symmetry** of $\sim$, we can say that also $b \sim a$; then, repeating the argument, we arrive at $[a] \subseteq [b]$. As we know,

$$([b] \subseteq [a]) \ \& \ ([a] \subseteq [b]) \implies [a] = [b].$$

(2) $\implies$ (3) is obvious **in view of** (6.8).

(3) $\implies$ (1). Let $c \in [a] \cap [b]$. Then

$$\left. \begin{array}{l} c \in [a] \xLeftrightarrow{\text{def}} a \sim c \\ c \in [b] \xLeftrightarrow{\text{def}} b \sim c \xRightarrow{\text{S}} c \sim b \end{array} \right\} \xRightarrow{\text{T}} a \sim b. \quad \square$$

Thus, equivalence classes are either equal or disjoint, and each element $a \in A$ belongs to an equivalence class, *viz.* $[a]$. Hence, distinct equivalence classes define a *partition* of A :

$$(6.10) \quad A = A_1 \cup A_2 \cup \dots, \quad A_i \cap A_j = \emptyset \quad \text{for } i \neq j.$$

The right way to think about this is that **objects** (elements of A) are distributed into **boxes** (equivalence classes). Sometimes, the **boxes** are labeled, *e.g.*,

- **red**, **blue**, **green**, **magenta**, $\dots$, or
- Mercedes, BMW, Toyota, $\dots$, or
- $0, 1, \dots, m-1$ (the residues mod m);

then, we speak about the **defining property**. Sometimes, there are no labels (like triangular shapes, or cardinalities); then, the property (label) is **defined** by the **objects** that the **box** contains!

Conversely, any partition (6.10) gives rise to an **equivalence relation**: we **define**

$$a \sim b \quad \text{iff} \quad a \text{ and } b \text{ are in the same } A_i \text{ in (6.10).}$$

Check that this **is** an equivalence relation!

Summarizing, **equivalence relations** and **partitions** are two ways to speak about the same thing. **Sometimes** (when we have a well-understood “defining property” like color, make, residue), it is more convenient to start with a **partition**. **In other situations** (when it is difficult to explain what the “property” is, like shape or infinite cardinality), we start with the **equivalence relation**. In either case, we should keep in mind that **we have and can use both equivalence relation and partition**.

Leaving the “general theory” aside, the two most important partitions encountered in this course are

- **congruence classes** of integers modulo a fixed $m \in \mathbb{Z}^+$;
- **cardinalities** of sets.

7. Partial orders

Definition 7.1. A relation $\preccurlyeq$ on a set S is called a *partial order* if it is

- *reflexive*, i.e., $a \preccurlyeq a$ for any $a \in S$,
- *anti-symmetric*, i.e., $a \preccurlyeq b$ and $b \preccurlyeq a$ imply $a = b$, and
- *transitive*, i.e., $a \preccurlyeq b$ and $b \preccurlyeq c$ imply $a \preccurlyeq c$.

The pair $(S, \preccurlyeq)$ is called a *partially ordered set*, or a *poset*. Usually, $\preccurlyeq$ is understood and we just speak about a *poset* S .

The Book uses *ordering* instead of *order*, which is not common.

A partial order does express out intuitive idea about an “ordering” of the elements of S except that, in general, we do not require that any two elements should be *compatible* (that is why “partial”). Usually, it is $\preccurlyeq$ that is given, but we use it to **define** a bunch of others:

$a \prec b$ means $a \preccurlyeq b$ but $a \neq b$,

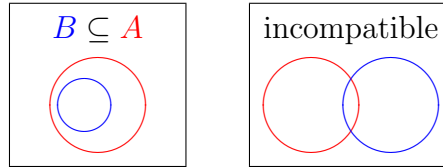
$a \succcurlyeq b$ means $b \preccurlyeq a$, **not the same as** $a \not\prec b$

$a \succ b$ means $b \prec a$. **not the same as** $a \not\prec b$

Note that, if $\preccurlyeq$ is a partial order, then so is $\succcurlyeq$ (reversing the order).

Example 7.2. The three most important examples are:

- (1) the usual *linear order* $\leq$ on $\mathbb{R}$ or subsets thereof, like $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\{1, 2, \dots, n\}$, etc.; it is *linear* as any two elements are *compatible*, see below;
- (2) the *inclusion* $\subseteq$ on the power set $\mathcal{P}(\mathcal{U})$ of any set $\mathcal{U}$; this one is truly *partial*:



Here, we can also use the symbols $\subset$, $\supseteq$, $\supset$;

- (3) the *divisibility* $|$ on the **positive** integers $\mathbb{Z}^+$ or subsets thereof; this one is also *partial*, as, e.g., 2 and 3 are incompatible: neither $2 \mid 3$, nor $3 \mid 2$. There are no analogues of $\prec$, $\succcurlyeq$, $\succ$ for this order.

As a remark, recall that, when we defined the *greatest common divisor* $d := \gcd(a, b)$, we said that it was “greatest” in all senses. Now, this claim can be explained. The set C of **positive** common divisors of a and b has (at least) two natural orders: the linear order $\leq$ and divisibility order $|$. So, d is the *greatest element* (see below) with respect to both. Note also that, for **positive** integers a, b ,

$$a \mid b \text{ implies } a \leq b$$

but, certainly, not *vive versa*.

Example 7.3. This is **not** a partial order on $\mathcal{P}(S)$:

$$A \preccurlyeq B \text{ iff } |A| \leq |B|.$$

Why?

We use $\preccurlyeq$ to denote a partial order. Particular partial orders usually have their own notation, like $\leq$, $\subseteq$, $|$, etc.

Prove this!

7.1. Linearly and well-ordered set.

Definition 7.4. As mentioned above, two elements a, b of a poset $(S, \preceq)$ are called *compatible* (*comparable* in the Book) if either $a \preceq b$ or $b \preceq a$. Otherwise, if neither $a \preceq b$ nor $b \preceq a$, the elements a, b are *incompatible* (*incomparable*).

A partial order $\preceq$ is called a *total* or *linear order* if any two elements are compatible. In this case, $(S, \preceq)$ is a *totally* or *linearly ordered set*.

An example of a *total order* is $\leq$. (See also the *lexicographic order* below.) Partial orders that are **not** total are, e.g., $\subseteq$ or $|$.

Definition 7.5. A *linearly ordered set* $(S, \preceq)$ is said to be *well ordered* if any non-empty subset of S has a least element.

For some reason,
there is no term
good order.

The examples for us are $(\mathbb{Z}^+, \leq)$ or $(\mathbb{N}, \leq)$.

We know how a well-ordering is used: to do induction! Here is a generalized induction principle.

Theorem 7.6 (well-ordered induction). Let $(S, \preceq)$ be a well-ordered set. Then, a statement $P(x)$ is true for any $x \in S$ if

Inductive step: for each $y \in S$, if $P(x)$ holds for all $x \in S$, $x \prec y$, then also $P(y)$ holds.

Proof. This is left as an exercise: consider the “bad” set

$$A := \{x \in S \mid P(x) \text{ does not hold}\},$$

assume that $A \neq \emptyset$, let y be the least element of A (which exists due to the well-ordering), and get a contradiction. $\square$

Note that the *basic step* is missing here! In fact, it is incorporated into the *inductive step*: if $y \in S$ is the least element of S , then

$$P(x) \text{ holds for all } x \in S, x \prec y$$

is **automatically true** just because there is no $x \prec y$ and, hence, there is nothing to check; thus, by the *inductive step*, $P(y)$ must hold.

7.2. The lexicographic order. The *lexicographic order* (aka *dictionary order*) is a certain canonical (partial) order defined on the Cartesian product of (partially) ordered sets or on the set of strings in a (partially) ordered alphabet.

Cartesian product. Assume that $(S_i, \preceq_i)$, $1 \leq i \leq n$, are posets, and let

$$S := S_1 \times \dots \times S_n.$$

Recall that this is the set of ordered n -tuples $(a_1, \dots, a_n)$. We **define**

$$(a_1, \dots, a_n) \prec (b_1, \dots, b_n)$$

if there exists an index $0 \leq p \leq n$ such that

$$a_1 = b_1, a_2 = b_2, \dots, a_p = b_p \quad \text{and} \quad a_{p+1} \prec_{p+1} b_{p+1}.$$

(Respectively, $\preceq$ is defined as $\prec$ or $=$.) In other words, *we compare the first distinct entries* of the two tuples.

Example 7.7. In $(\mathbb{Z}, \leq)^3$:

$$(1, 2, 2) \prec (2, 1, 1), \quad (1, 1, 1) \prec (1, 1, 2), \quad (1, 1, 1) \preceq (1, 1, 1) \quad \text{as } (1, 1, 1) = (1, 1, 1)$$

In $(\mathbb{Z}^+, |)^2$:

$$(2, 3) \prec (4, 2), \quad (2, 3) \text{ and } (3, 3) \text{ are incompatible.}$$

Strings. For strings (which are nothing but tuples of characters), the procedure is similar except that, in addition, we need to take into account the fact that the two lengths may differ. Thus, we define $a \prec b$ if there is a $p \geq 0$ such that

$$a_1 = b_1, \dots, a_p = b_p \quad \text{and} \quad \begin{cases} a_{p+1} \prec b_{p+1} & \text{or} \\ \text{length}(a) = p < \text{length}(b). \end{cases}$$

Alternatively, we can pretend that the shorter string is padded to the common length n by an **extra** character $\perp$ **that is smaller than anything else**, and then the two strings are compared as n -tuples. This is the word order commonly used in the dictionaries.

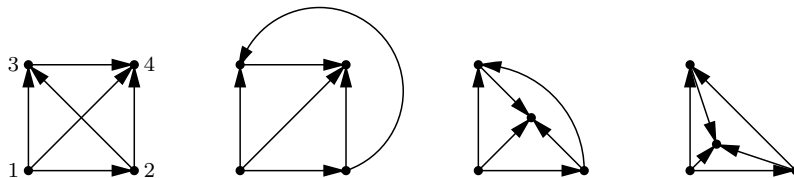
Example 7.8. We have

$$\begin{aligned} \text{discreet} &\prec \text{discrete} \\ \text{discreet}_{\perp\perp\perp} &\prec \text{discreetness} \\ \text{discrete} &\prec \text{discretion} \end{aligned}$$

In the last example, there is no need in the padding because a distinct character is found before the shorter string ends.

8. Hasse diagrams

In general, a binary relation R on a finite set S can be represented by a *directed graph* (aka *digraph*, or just *graph* if “directed” is understood), whatever this means. Roughly, we represent each element $a \in S$ by a vertex (a dot) and, whenever $a R b$, draw an arrow $\overrightarrow{ab}$ from a to b . We pay **no attention** to the mutual position of the vertices, shape or length of the arrows or whether they intersect or not. These are the “same” graph:



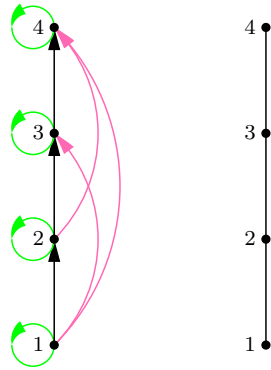
Accidentally, the corresponding relation is

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \subset \{1, 2, 3, 4\}^2,$$

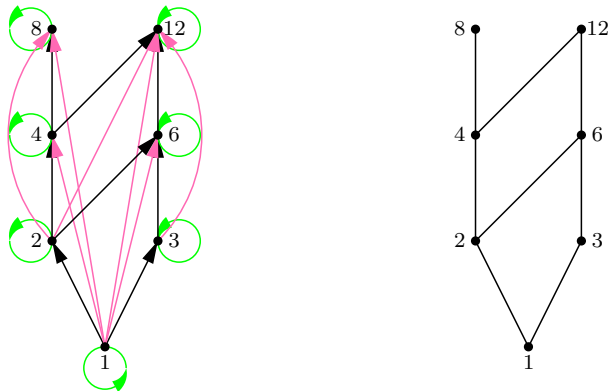
i.e., the usual **strict** order $<$ on $\{1, 2, 3, 4\}$. (This is **not** a partial order by our definition, as it is not reflexive.)

However, if we do this to a “typical” partial order, the result is going to be extremely messy and incomprehensible. This is the relation as simple as the usual linear order $\leq$ on $\{1, 2, 3, 4\}$ (we draw the graph slightly differently):

Disregarding the green circles, we get the “same” graph as above!



Or, consider the divisibility order $\mid$ on $\{1, 2, 3, 4, 6, 8, 12\}$:



In both cases, we see that a lot of arrows are *unnecessary* because *they follow from the properties of partial orders*. Therefore, we adopt a number of extra **conventions**:

- (1) By the **reflexivity**, we know that there is a **loop** at each vertex. Since we **know** that they are there, there is **no need to draw them** explicitly; we can just **assume** them!
- (2) A lot of **arrows** are there just because of the **transitivity**: if there are arrows $\overrightarrow{ab}$ and $\overrightarrow{bc}$, automatically there also is an arrow $\overrightarrow{ac}$. Again, there is no need to draw such **superfluous arrows** explicitly, and we erase them.
- (3) Since the relation is **anti-symmetric**, for two **distinct** vertices $a \neq b$, we cannot have both arrows $\overrightarrow{ab}$ and $\overrightarrow{ba}$. It follows that we can position the vertices so that all arrows point upwards (as in the pictures). Once this is done, we can **erase the arrowheads**.

This needs proof!
To be done later,
see §9.2

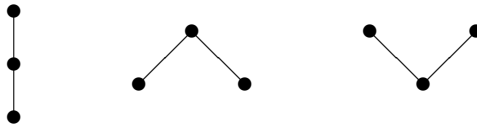
The resulting graph (which looks *undirected*, but *is directed implicitly*), is called the *Hasse diagram* of the given partial order. **From now on, we always depict partial orders by their *Hasse diagrams*.**

In a **partial** order, an *immediate successor* is **not** necessarily unique!

Remark 8.1. In **item 2**, we can say that we only draw an arrow $\overrightarrow{ab}$ if b is an *immediate successor* of a , meaning that $a \prec b$ and there is no $c \in S$ such that $a \prec c$ and $c \prec b$ (there is no c between a and b).

In **item 3**, we need (and have) a bit more: by the **anti-symmetry** and **transitivity**, the original graph has no *oriented cycles*: you cannot start at a point a , follow the arrows, and come back to a .

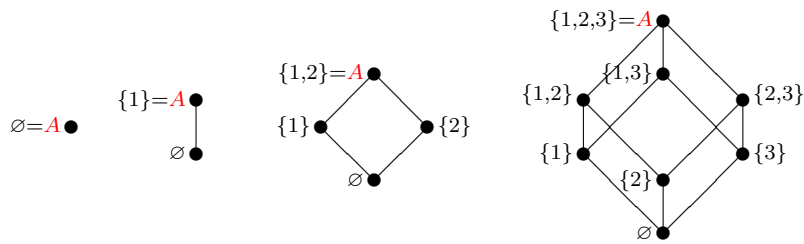
Remark 8.2. With *Hasse diagrams*, it is no longer quite true that we do not care about the mutual position of the vertices. Up/down **does** matter (or else we would have to retain the arrowheads). Thus, these are **different** *Hasse diagrams*, representing quite different posets:



Example 8.3. The *Hasse diagram* of a *linearly ordered set* is a *linear tree*:



In this sense, *linear orders* on finite sets are boring. Here are the *Hasse diagrams* of the power set $(\mathcal{P}(A), \subseteq)$ of an n -element set $A := \{1, \dots, n\}$, $n = 0, 1, 2, 3$:



In general, the *Hasse diagram* of $(\mathcal{P}(A), \subseteq)$ is (an appropriate projection of) the *n -dimensional cube*.

9. Maximal, greatest, *etc.* elements

In the rest of this lecture(s), we mainly introduce a lot of terminology related to posets. Since the order is **partial**, some of the terms are a bit **counterintuitive**.

Definition 9.1. Consider a poset $(S, \preceq)$.

An element $b \in S$ is called **maximal** (**minimal**) if there is no element $c \in S$ such that $b \prec c$ (respectively, $b \succ c$).

An element $a \in S$ is called **greatest** (**least**) if $a \succcurlyeq c$ (respectively, $a \preccurlyeq c$) for any element $c \in S$.

Warning 9.2. Pay close attention: in general,

$$\text{maximal} \neq \text{greatest}, \quad \text{minimal} \neq \text{least}.$$

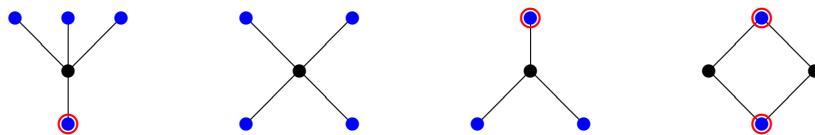
Example 9.3. In the language of *Hasse diagrams*,

- **maximal** means that **nothing is above it**, whereas
- **greatest** means **above everything else**,

where “above” means “can be reached by following the arrows.”

Of course, the same applies to **minimal/least**/below.

Here are examples.



Example 9.4. In $(\mathcal{P}(A), \subseteq)$, the only **minimal** = **least** element is $\emptyset$, and the only **maximal** = **greatest** element is A itself.

In $(\mathbb{Z}^+, |)$, the only **minimal** = **least** element is 1, and there are no **maximal** or **greatest** elements.

These statements are suggested by the examples.

Theorem 9.5. A **greatest** (**least**) element, **if exists**, **is unique**.

Any **greatest** (**least**) element is **maximal** (**minimal**), but not vice versa.

A **nonempty finite poset** has at least one **maximal** (**minimal**) element.

Moreover, any $c \in S$ is bounded from above (below) by some **maximal** (respectively, **minimal**) element.

A **finite poset** has **greatest** (**least**) element if and only if it has a **unique maximal** (respectively, **minimal**) element.

Proof. This is what we call “general nonsense.” We do **maximal/greatest**.

If $a_1, a_2 \in S$ are both greatest, then

$$\left. \begin{array}{l} a_1 \succcurlyeq a_2 \quad (\text{since } a_1 \text{ is greatest}) \\ a_2 \succcurlyeq a_1 \quad (\text{since } a_2 \text{ is greatest}) \end{array} \right\} \xRightarrow{\text{AS}} a_1 = a_2.$$

If $a \succcurlyeq c$ for **any** $c \in S$, there is no $b \succ a$.

Consider $b_1 := c \in S$. If it is maximal, we are done. Otherwise, there is $b_2 \succ b_1$. If b_2 is maximal, we are done; if not, there is $b_3 \succ b_2$. Continuing, we obtain a chain

$$c = b_1 \prec b_2 \prec \dots \prec b_k$$

of necessarily pairwise distinct elements. Since the set is finite, it must terminate at some step b_k , meaning that we cannot find $b_{k+1} \succ b_k$. This means that b_k is maximal.

We have $a \succcurlyeq b$ for any $b \in S$. If b is **maximal** (no $a \succ b$), then $b = a$. Thus, all maximal are equal to the unique greatest. Conversely, any $c \in S$ is bounded from above by **the** unique **maximal** element b : $b \succcurlyeq c$. Hence, b is greatest. $\square$

The last two statements are not true in infinite posets

9.1. **Lattices.** Let $(S, \preccurlyeq)$ be a poset and $A \subseteq S$ a subset.

An *upper bound* of A is an element $u \in S$ such that $a \preccurlyeq u$ for each $a \in A$. If u is an *upper bound* and $u \preccurlyeq u'$, then u' is also an *upper bound*. (Note that we do **not** insist that $u \in A$.) It *may happen* that the set of all *upper bounds* for A has a (necessarily unique) *least element*; then, this least element is called the *least upper bound* of A :

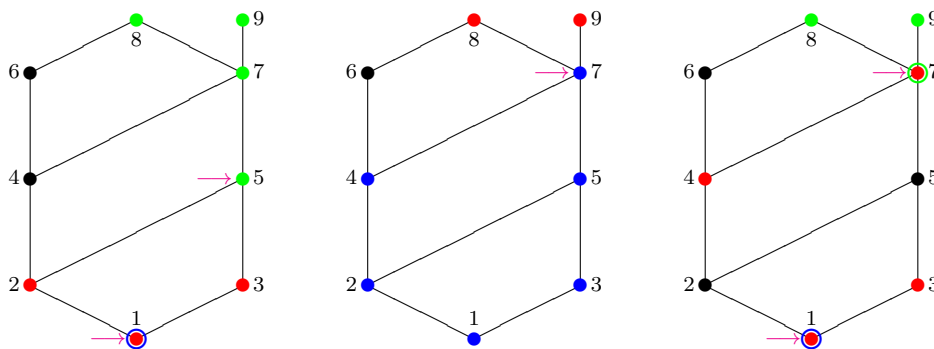
$$\text{lub } A = \sup A.$$

Again, we do **not** insist that $\sup A \in A$.

Similarly, we define the set of *lower bounds* of A and, if exists, the *greatest lower bound*

$$\text{glb } A = \inf A.$$

Example 9.6. Find the (*least*) *upper bound(s)* and (*greatest*) *lower bound(s)* for the subset $A := \{1, 2, 3\}$, $\{8, 9\}$, or $\{1, 3, 4, 7\}$ in $(S, \preccurlyeq)$:



A poset $(S, \preccurlyeq)$ is called a *lattice* if each **pair** $\{a, b\} \subseteq S$ has the *least upper bound* and *greatest lower bound*.

Example 9.7. Any totally ordered set is a lattice.

Example 9.8. The power set $(\mathcal{P}(\mathcal{U}), \subseteq)$ (for any universe $\mathcal{U}$) is a lattice:

$$\text{lub}\{A, B\} = A \cup B, \quad \text{glb}\{A, B\} = A \cap B.$$

In fact, **any** (not just 2-element) subset $\mathcal{S} \subseteq \mathcal{P}(\mathcal{U})$ has lub and glb:

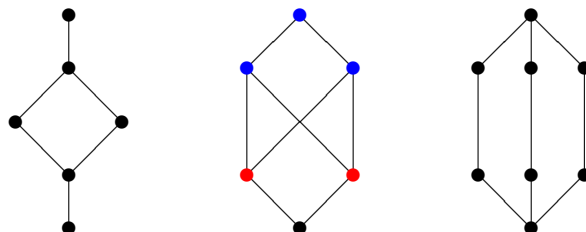
$$\text{lub } \mathcal{S} = \bigcup_{A \in \mathcal{S}} A, \quad \text{glb } \mathcal{S} = \bigcap_{A \in \mathcal{S}} A.$$

Likewise, $(\mathbb{Z}^+, |)$ is a lattice:

$$\text{lub}\{a, b\} = \text{lcm}(a, b), \quad \text{glb}\{a, b\} = \text{gcd}(a, b).$$

In fact, any subset has $\text{glb} = \text{gcd}$, and any **finite** subset has $\text{lub} = \text{lcm}$.

Example 9.9. Which of the following are lattices:



9.2. The topological sorting algorithm.

Problem 9.10. Given a poset $(S, \preceq)$, find a **linear** order on S *extending* $\preceq$, i.e., a **linear** order $\leq$ such that

$$a \preceq b \text{ implies } a \leq b.$$

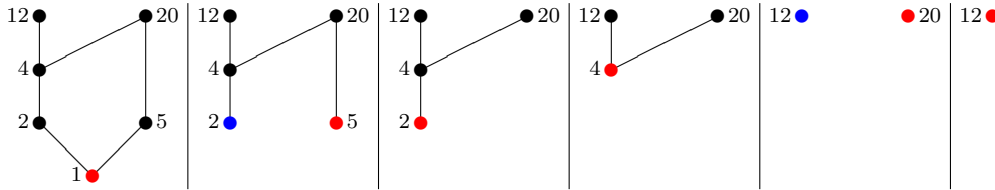
Solution (the topological sorting algorithm). On a **finite** set, this can always be done. We start with $S_1 := S$, and proceed step by step, constructing consecutive sets S_k and writing consecutive elements a_k ; the order

$$a_1 < a_2 < \dots < a_k < \dots$$

in which these elements are written is the linear order we are looking for. The step of the algorithm is very simple:

Step k : if $S_k = \emptyset$, stop: we are done. Otherwise, choose a minimal element $a_k \in S_k$ and let $S_{k+1} := S_k - \{a_k\}$. Go to the next step $k+1$.

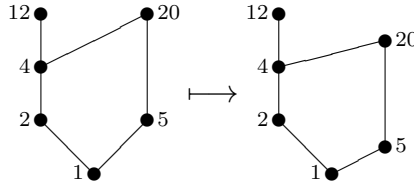
Example 9.11. Let $S := \{1, 2, 4, 5, 12, 20\}$ and $\preceq = |$. Of course, we know **one** linear order extending $|$, viz. the natural order $\leq$. But in our algorithm there are some choices, so let us try to find another one:



Here, the red vertices represent the minimal element chosen at each step. The resulting order is

$$1 < 5 < 2 < 4 < 20 < 12.$$

Intuitively, we move the vertices *slightly* so that no two are at the same height, and then we just order them according to the height:



Remark 9.12. The existence of a linear order extending the given partial order justifies our usage of the Hasse diagrams: to draw the diagram, *we need to arrange the height of the vertices appropriately!* Now, we know that this can be done!

APPENDIX A. Sample problems

Problem A.1 (No. 1). Consider *ternary strings*, i.e., strings in the 3-digit alphabet $\{0, 1, 2\}$. Let c_n be the sequence counting the number of strings of length $n \geq 0$ without consecutive 1's or consecutive 2's. Find a recurrence relation for c_n .

Solution: A string s of length n is obtained by appending a digit 0, 1, or 2 to a string s' of length $n - 1$. However, the constraints given limit our options depending on the last digit of s' :

$$s = \begin{cases} s''0\{0, 1, 2\} \\ s''1\{0, 2\} \\ s''2\{0, 1\} \end{cases}$$

The number of $(n-1)$ -strings ending in 0 is c_{n-2} (we can append 0 to **any** string), and each of them gives us 3 choices. Hence, by the **Subtraction Rule**, the number of $(n-1)$ -strings **not** ending in 0 is $c_{n-1} - c_{n-2}$, and each of these gives us 2 choices. Therefore,

$$c_n = 3c_{n-2} + 2(c_{n-1} - c_{n-2}) = 2c_{n-1} + c_{n-2}.$$

The initial conditions are $c_0 = 1$, $c_1 = 3$.

Although this is not required, let us also try to solve this relation. The characteristic equation is

$$r^2 = 2r + 1, \quad \text{hence } r = 1 \pm \sqrt{2}.$$

We have two distinct real roots, and the general solution is

$$c_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n.$$

Substituting $n = 0, 1$, we get

$$A + B = 1, \quad (A + B) + (A - B)\sqrt{2} = 3, \quad \text{or } A - B = \sqrt{2}.$$

Solving,

$$A = \frac{1 + \sqrt{2}}{2}, \quad B = \frac{1 - \sqrt{2}}{2}$$

Hence,

$$c_n = \frac{1}{2}((1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}).$$

Problem A.2 (No. 5). How many 6-card hands from a standard deck of 52 cards have at least one card of each suit?

Solution: We use the *principle of inclusion/exclusion*. The “bad” conditions are

$$P_i: \text{ suit } i \text{ is missing in the hand, } i = 1, 2, 3, 4.$$

A hand satisfying k of these conditions is any hand obtained by dealing a deck with k suits missing, *i.e.*,

$$N_k := N(P_{i_1} \dots P_{i_k}) = C(52 - 13k, 6);$$

this does not depend on the particular conditions but only on their number. Thus, the number is

$$\begin{aligned} \# &= N_0 - C(4, 1)N_1 + C(4, 2)N_2 - C(4, 3)N_3 + N_4 \\ &= C(52, 6) - 4C(39, 6) + 6C(26, 6) - 4C(13, 6) + C(0, 6) \\ &= \boxed{8682544}. \end{aligned}$$

Problem A.3. What is wrong in this “solution” to the previous problem: first, deal one card of each suit, in 13^4 ways, then deal two more cards from the rest, in $C(48, 2)$ ways, to obtain

$$13^4 \cdot C(48, 2) = 32216808?$$

Problem A.4 (No. 7). How many permutations of the 26 letters of the English alphabet do not contain **math**, **path**, **sum**, or **theory**?

Solution: We use the *principle of inclusion/exclusion*. Introduce the “bad” conditions:

$$\begin{aligned} P_1(s): & \quad s \text{ contains } \text{math}, \\ P_2(s): & \quad s \text{ contains } \text{path}, \\ P_3(s): & \quad s \text{ contains } \text{sum}, \\ P_4(s): & \quad s \text{ contains } \text{theory}. \end{aligned}$$

Then (be careful!):

$$\begin{aligned} N &= 26!, & (\text{all strings}) \\ N(P_1) &= 23!, & (\text{math is a single letter}) \\ N(P_2) &= 23!, & (\text{path is a single letter}) \\ N(P_3) &= 24!, & (\text{sum is a single letter}) \\ N(P_4) &= 21!, & (\text{theory is a single letter}) \\ N(P_1 P_2 \dots) &= N(P_1 P_2) = 0, & (\text{cannot have math and path}) \\ N(P_1 P_3) &= 21!, & (\text{sumath is a single letter}) \\ N(P_1 P_4) &= 19!, & (\text{matheory is a single letter}) \\ N(P_2 P_3) &= 21!, & (\text{path and sum are single letters}) \\ N(P_2 P_4) &= 19!, & (\text{patheory is a single letter}) \\ N(P_3 P_4) &= 19!, & (\text{sum and theory are single letters}) \\ N(P_1 P_3 P_4) &= 17!, & (\text{sumatheory is a single letter}) \\ N(P_2 P_3 P_4) &= 17!, & (\text{sum and patheory are single letters}) \end{aligned}$$

Therefore, the number in question equals

$$\begin{aligned} \# &= 26! - 23! - 23! - 24! - 21! \\ &+ 0 + 21! + 19! + 21! + 19! + 19! \\ &- 0 - 0 - 17! - 17! + 0 \\ &= \boxed{26! - 24! - 2 \cdot 23! + 21! + 3 \cdot 19! - 2 \cdot 17!} \quad (\text{best to leave it in this form}) \end{aligned}$$

Problem A.5 (No. 10). Find the generating function for the number of integer solutions to

$$2w + 3x + 5y + 7z = n, \quad w, x, y, z \geq 0.$$

Solution: We follow the standard paradigm. Each of the four terms gets its own generating function, [according to values that this term may take](#):

$$2w: \quad G_w(t) = t^0 + t^2 + t^4 + \dots = \sum_{n=0}^{\infty} t^{2n} = \frac{1}{1-t^2},$$

$$3x: \quad G_x(t) = t^0 + t^3 + t^6 + \dots = \sum_{n=0}^{\infty} t^{3n} = \frac{1}{1-t^3},$$

$$5y: \quad G_y(t) = t^0 + t^5 + t^{10} + \dots = \sum_{n=0}^{\infty} t^{5n} = \frac{1}{1-t^5},$$

$$7z: \quad G_z(t) = t^0 + t^7 + t^{14} + \dots = \sum_{n=0}^{\infty} t^{7n} = \frac{1}{1-t^7}.$$

(We use t for the dummy variable to avoid mixing it with the unknown x .) Then, the generating function for the sequence counting the number of solutions is the product of the above:

$$G(t) = G_w(t)G_x(t)G_y(t)G_z(t) = \boxed{\frac{1}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}}.$$

MARGINAL NOTES

- 1/ 4: This is the formal *definition* of a function: a subset of $A \times B$ with certain properties
- 2/ 11: This is the so-called *transitive closure* of R
- 3/ 12: We discussed this at the time!
- 4/ 12: Prove this!
- 5/ 12: Here $|a|$ is the length and a_m is the truncation of a to the first m characters
- 6/ 13: This is a common argument: the equivalence classes appear as the *orbits* of a *group action*
- 7/ 13: Another common thing: *define* a property by *listing* the objects that have it! Only mathematicians can do that!
- 8/ 13: Well, this is **too** much of formal logic!
- 9/ 16: We use $\preceq$ to denote a partial order. Particular partial orders usually have their own notation, like $\leq$, $\subseteq$, $|$, *etc.*
- 10/ 16: Prove this!
- 11/ 17: For some reason, there is no term *good order*.
- 12/ 20: Disregarding the green circles, we get the “same” graph as above!
- 13/ 20: This needs proof! To be done later, see §9.2
- 14/ 20: In a **partial** order, an *immediate successor* is **not** necessarily unique!
- 15/ 22: The last two statements are not true in infinite posets