Math 132, Spring 2005

Solutions to Midterm 1

Problem 1. Find the number of ways to place $2n+1$ indistinguishable balls in three boxes so that the 'triangle inequality' holds: any two boxes together contain more balls than the third box.

SOLUTION: To satisfy the inequality, each box should receive less than a half, *i.e.*, at most n balls. The number of all placements is the number of non-negative integral solutions to the equations $n_1 + n_2 + n_3 = 2n+1$, which equals $C((2n+1)+3-1, 2) = (2n+3)(2n+2)/2$. The number of bad placements, when the first box receives $n_1 = n+1+m_1 > n$ balls, is the number of solutions (again in non-negative integers) to the equation $m_1 + n_2 + n_3 = n$, which equals $C(n+3-1, 2) = (n+2)(n+1)/2$. The same applies to the second and the third boxes receiving more than n balls, and the latter three cases are mutually exclusive. Hence, the number in question is

$$\text{(all cases)} - \text{(bad cases)} = \frac{(2n+3)(2n+2)}{2} - 3\frac{(n+2)(n+1)}{2} = \boxed{\frac{n(n+1)}{2}}.$$

Problem 2. Does the following hold for all statements p, q, r?

$$(p \to q) \wedge (q \to r) \wedge (r \to p) \iff \neg\big(\neg(p \wedge q \wedge r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r)\big)$$

SOLUTION: Applying DeMorgan laws to the right hand side, we get $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$. Now, express the left hand side in terms of $\neg$, $\wedge$, $\vee$ and 'multiply out':

$$(\neg p \vee q) \wedge (\neg q \vee r) \wedge (\neg r \vee p) \iff (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r) \vee \dots,$$

where the other six terms are triple conjunctions containing a variable and its negation and thus identically false. Comparing this to the previous result, we find that the equivalence **does hold**.

Remark. In plain English, the right hand side (after the simplification) means that p, q, and r are equivalent, *i.e.*, either all three true or all three false, and the left hand side is a standard way to prove such an equivalence.

Problem 3. Show that for any integer $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

SOLUTION: The basis step ($n=2$): $1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{2}{\sqrt{2}} = \sqrt{2}$.

The inductive step: assume that the statement holds for $n = k$ and consider $n = k+1$:

$$\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} > \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}.$$

(Here, for the first inequality we use the induction assumption, and for the second one, $k+1 > k$.)

Problem 4. Show that for any positive integer n one has g.c.d.$(12n+7, 7n+4) = 1$.

SOLUTION: We mimic the Euclidean algorithm:

$$\begin{aligned}
12n+7 &= (7n+4)\cdot 1 + (5n+3),\\
7n+4 &= (5n+3)\cdot 1 + (2n+1),\\
5n+3 &= (2n+1)\cdot 2 + (n+1),\\
2n+1 &= (n+1)\cdot 2 + (-1) \qquad \text{(last nonzero 'remainder')}.
\end{aligned}$$

Problem 5. How many words of length $m > 0$ can be made out of four letters **A**, **B**, **C**, **D** so that each letter is used at least once?

SOLUTION: We are speaking about maps from an m-element set ('places' in a word) to the $n = 4$ element set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. (The map assigns to a place the letter that goes to that place.) The condition that each letter must be used means that the map must be onto. Thus, just apply the formula for the number of onto maps:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m = \boxed{4^m - 4\cdot 3^m + 6\cdot 2^m - 4}.$$

(You can check that this formula does give zero for $m = 1, 2, 3$.)