Solutions to Final Exam

Problem 1. Evaluate:

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{2}{\pi}\tan^{-1}3n\right)^n$.

(b) $\displaystyle\lim_{x\to 0}\frac{x-\tan^{-1}x}{\sin x - x}$.

SOLUTION: (a) Denote the limit in question by A (and assume n real, so that l'Hôpital's rule can be used). Then

$$\ln A = \lim_{n\to\infty}\frac{\ln\frac{2}{\pi}\tan^{-1}3n}{1/n} = \left[\frac{0}{0}\right] = \lim_{n\to\infty}\frac{\dfrac{3}{1+(3n)^2}}{\tan^{-1}3n\cdot(-1/n^2)} = -\frac{2}{3\pi};\quad \text{hence, } A = \boxed{e^{-\frac{2}{3\pi}}}.$$

(b) Use power series:

$$\lim = \lim_{x\to 0}\frac{x-\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots\right)}{\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\dots\right)-x} = \lim_{x\to 0}\frac{\frac{1}{3}x^3-\frac{1}{5}x^5+\dots}{-\frac{1}{6}x^3+\frac{1}{120}x^5-\dots} = \boxed{-2}.$$

(Of course, the same can be done by repeating l'Hôpital's rule several times.)

Problem 2. (a) Find the series' radius and interval of convergence. Then identify the values of x for which the series converges absolutely/converges conditionally/diverges:

$$\sum_{n=2}^{\infty}\frac{(x-1)^n}{n\ln n}.$$

(b) Find the Taylor series generated by the function $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2$ at $x=-1$.

SOLUTION: (a) One has

$$\frac{1}{R} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = 1\cdot\lim_{n\to\infty}\frac{n+1}{n} = 1.$$

(In the second limit, we used l'Hôpital's rule.) Thus, $R=1$ and the interval of convergence is $|x-1|<1$, *i.e.*, $0<x<2$. At $x=2$, the series has positive terms and diverges by the integral test, as

$$\int_2^\infty \frac{dx}{x\ln x} = \int_2^\infty\frac{d(\ln x)}{\ln x} = \ln\ln x\Big|_2^\infty = \infty.$$

At $x=0$, the series is alternating and converges by the Leibnitz theorem (check it!), whereas the corresponding series of absolute values is the same power series at $x=2$, *i.e.*, it diverges. Thus, the series:

$\boxed{\text{converges absolutely for } x\in(0,2)\text{, converges conditionally for } x=0\text{, and diverges otherwise}}$.

(b) Just compute the derivatives and evaluate them at $x=-1$:

$$\begin{aligned}
f(x) &= 3x^5 - x^4 + 2x^3 + x^2 - 2, & f(-1) &= -7,\\
f'(x) &= 15x^4 - 4x^3 + 6x^2 + 2x, & f'(-1) &= 23,\\
f''(x) &= 60x^3 - 12x^2 + 12x + 2, & f''(-1) &= -82,\\
f'''(x) &= 180x^2 - 24x + 12, & f'''(-1) &= 216,\\
f^{\text{IV}}(x) &= 360x - 24, & f^{\text{IV}}(-1) &= -384,\\
f^{\text{V}}(x) &= 360, & f^{\text{V}}(-1) &= 360.
\end{aligned}$$

(All other derivatives are identically zero.) Thus,

$$\begin{aligned}
f(x) &= -7 + \frac{23}{1}(x+1) - \frac{82}{2}(x+1)^2 + \frac{216}{6}(x+1)^3 - \frac{384}{24}(x+1)^4 + \frac{360}{120}(x+1)^5\\
&= \boxed{-7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5}.
\end{aligned}$$

Problem 3. Find the first six terms (up to x^5) of the Maclaurin series of the solution to the initial value problem $y'' + xy' + e^x y = \sin x,\ y(0) = 1,\ y'(0) = 0.$

SOLUTION: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$. Then $a_0 = 1$, $a_1 = 0$ (from the initial data), and one has

$$\begin{aligned} y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots, \\ xy' &= a_1x + 2a_2x^2 + 3a_3x^3 + \dots, \\ y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots, \\ e^x y &= a_0 + (a_1 + a_0)x + \Big(a_2 + a_1 + \frac{a_0}{2}\Big)x^2 + \Big(a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6}\Big)x^3 + \dots, \\ \sin x &= x - \frac{1}{6}x^3 + \dots. \end{aligned}$$

Substituting (and cutting at x^4), we get

$$(2a_2 + a_0) + (6a_3 + 2a_1 + a_0 - 1)x + \Big(12a_4 + 3a_2 + a_1 + \frac{a_0}{2}\Big)x^2 + \Big(20a_5 + 4a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6} + \frac{1}{6}\Big)x^3 = 0$$

and, equating the coefficients of similar powers of x and solving the equations consecutively,

$$a_2 = -\frac{1}{2}a_0 = -\frac{1}{2}, \qquad a_3 = \frac{1}{6}(1 - a_0) - \frac{1}{3}a_1 = 0, \qquad a_4 = -\frac{1}{24}a_0 - \frac{1}{12}a_1 - \frac{1}{4}a_2 = \frac{1}{12},$$
$$a_5 = -\frac{1}{120} - \frac{a_0}{120} - \frac{a_1}{40} - \frac{a_2}{20} - \frac{a_3}{5} = \frac{1}{120}.$$

Finally, $\boxed{y = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \dots}$.

Alternatively, you could have expressed $y'' = -xy' - e^x y + \sin x$, differentiated **first** three more times (**using the appropriate rules**, like the product rule), **then** substituted the initial conditions to find consecutively $y''(0) = -1$, $y'''(0) = 0$, $y^{\mathrm{IV}}(0) = 2$, and $y^{\mathrm{V}}(0) = 1$, and, finally, used the Maclaurin formula.

Problem 4. (a) Represent the integral by a series: $\displaystyle\int_0^1 x \sin x^3\, dx.$

(b) Evaluate: $\displaystyle\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \dots + (-1)^{n-1}\frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \dots.$

SOLUTION: (a)

$$\int_0^1 x\sin x^3\, dx = \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}\Big) dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{6n+4}}{(2n+1)!}\, dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+5)(2n+1)!}}.$$

(b) The series in question is the Maclaurin series of $\tan^{-1} x$ evaluated at $x = 1/\sqrt{3}$. (Note that this point does belong to the interval of convergence of the series.) Hence, the sum is $\tan^{-1}\frac{1}{\sqrt{3}} = \boxed{\frac{\pi}{6}}$.

Problem 5. (a) Find all unit vectors $\mathbf{v}$ orthogonal to $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and lying in the plane spanned by $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(b) Find an equation of the plane containing the line L_1 given by $x = t + 6$, $y = 3t + 2$, $z = 5t + 4$, $t \in \mathbb{R}$, and parallel to the line L_2 given by $x = t + 3$, $y = -2t$, $z = -t + 9$, $t \in \mathbb{R}$. Find the distance from this plane to L_2.

SOLUTION: (a) One has $\mathbf{v} = s\mathbf{b} + t\mathbf{c}$ for some $s, t \in \mathbb{R}$ and $v \cdot \mathbf{a} = 0$. Hence, $s\mathbf{b} \cdot \mathbf{a} + t\mathbf{c} \cdot \mathbf{a} = 0$, or $s - t = 0$, *i.e.*, $s = t$ and $\mathbf{v} = t(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Normalizing, $\mathbf{v} = \boxed{\pm(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})/\sqrt{14}}$.

Alternatively, you could have normalized the vector $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$, which satisfies all other conditions. (Why?)

(b) The lines are defined by the vectors $\mathbf{v}_1 = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{v}_2 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, and for a normal vector to the plane in question one can take $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2 = 7\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$. The plane passes through any point of L_1, *e.g.*, through $(6, 2, 4)$. Hence, the equation is $7(x - 6) + 6(y - 2) - 5(z - 4) = 0$, or $\boxed{7x + 6y - 5z - 34 = 0}$. For the distance, one can compute the distance to the plane from **any** point of L_2, *e.g.*, from $(3, 0, 9)$. Using the formula, the result is $|7 \cdot 3 + 6 \cdot 0 - 5 \cdot 9 - 34|/\sqrt{7^2 + 6^2 + 5^2} = \boxed{58/\sqrt{110}}$.