Solutions to Midterm I

Problem 1. Find the limits:

(a) $\lim_{x\to 0}(\cos x)^{\frac{x}{x-\sin x}}$.

(b) $\lim_{x\to\infty}\frac{\cosh x + f(x)}{\sinh x - f(x)}$, assuming that $f(x)$ grows slower than e^x when $x \to \infty$.

SOLUTION: (a) Denote the limit in question by A. Then

$$\ln A = \lim_{x\to 0}\frac{x\ln\cos x}{x-\sin x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\ln\cos x - x\tan x}{1-\cos x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{-\tan x - \tan x - x\sec^2 x}{\sin x} = -3$$

(cancelling $\sin x$ and using $\lim_{x\to 0}(x/\sin x) = 1$). Hence, $A = \boxed{e^{-3}}$.

(b) Spell out and divide by e^x:

$$\lim_{x\to\infty}\frac{\frac{1+e^{-2x}}{2} + \frac{f(x)}{e^x}}{\frac{1-e^{-2x}}{2} - \frac{f(x)}{e^x}} = \boxed{1} \qquad \left(\text{using } \lim_{x\to\infty} e^{-2x} = 0 \text{ and } \lim_{x\to\infty}\frac{f(x)}{e^x} = 0\right).$$

Problem 2. Evaluate $\int \frac{x}{x^3+4x^2+8x+8}\,dx$.

SOLUTION: Note that $x = -2$ is a root of the denominator; then, dividing by $(x+2)$, we get $x^3+4x^2+8x+8 = (x+2)(x^2+2x+4)$. The partial fraction expansion is

$$\frac{x}{x^3+4x^2+8x+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+4}, \quad \text{or} \quad x = A(x^2+2x+4) + (Bx+C)(x+2).$$

Letting $x = -2$, we get $A = -\frac{1}{2}$. Letting $x = 0$, we get $0 = 4A + 2C$, or $C = 1$. Finally, equating the coefficients of x^2 (the highest power of x), we get $0 = A + B$, or $B = \frac{1}{2}$. In addition, notice that $x^2+2x+4 = (x+1)^2+3$ (complete the square) and $\frac{1}{2}x + 1 = \frac{1}{2}(x+1) + \frac{1}{2}$. Hence, the integral in question is[1]

$$-\frac{1}{2}\int\frac{dx}{x+2} + \frac{1}{2}\int\frac{x+1}{x^2+2x+4}\,dx + \frac{1}{2}\int\frac{dx}{(x+1)^2+3} = \boxed{-\frac{1}{2}\ln|x+2| + \frac{1}{4}\ln(x^2+2x+4) + \frac{\sqrt{3}}{6}\tan^{-1}\frac{x+1}{\sqrt{3}}}.$$

Problem 3. Evaluate:

(a) $\int \sinh\big(\ln(x+1) - \ln(x-1)\big)\,dx$.

(b) $\int x\sinh^{-1}(x^2)\,dx$.

SOLUTION: (a) First, simplify:

$$\sinh\big(\ln(x+1)-\ln(x-1)\big) = \sinh\ln\frac{x+1}{x-1} = \frac{1}{2}e^{\ln\frac{x+1}{x-1}} - \frac{1}{2}e^{-\ln\frac{x+1}{x-1}} = \frac{1}{2}\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right) = \frac{1}{x-1} + \frac{1}{x+1}.$$

Then the integral is $\ln|x-1| + \ln|x+1| = \boxed{\ln|x^2-1|}$.

(b) First, substitute $t = x^2$, $dt = 2x\,dx$. Then the integral in question is

$$\frac{1}{2}\int\sinh^{-1}t\,dt = \begin{bmatrix} u = \sinh^{-1}t & du = \frac{dt}{\sqrt{t^2+1}} \\ v = t & dv = dt \end{bmatrix} = \frac{1}{2}t\sinh^{-1}t - \frac{1}{2}\int\frac{t\,dt}{\sqrt{t^2+1}} = \boxed{\frac{1}{2}x^2\sinh^{-1}(x^2) - \frac{1}{2}\sqrt{x^4+1}}.$$

[1] I am skipping the $+C$ part to save space

Problem 4. (a) Find $g\left(\frac{\pi}{6}\right)$, where $g(x) = \frac{d}{dx}\int_{-\tan x}^{\tan x} \frac{dt}{(t^2+1)^2}\,dt$.

(b) Express $\operatorname{sech}^{-1}\left(\frac{1}{\sqrt{2}}\right)$ in terms of the natural logarithm.

SOLUTION: (a) Use the fundamental theorem of calculus and the chain rule:

$$g(x) = \frac{1}{(\tan^2 x+1)^2}(\tan x)' - \frac{1}{(\tan^2 x+1)^2}(-\tan x)' = 2\cos^2 x.$$

Then $g\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{\frac{3}{2}}$.

(b) Let $x = \operatorname{sech}^{-1}\left(\frac{1}{\sqrt{2}}\right)$. Then $\cosh x = \sqrt{2}$, *i.e.*, $e^x + e^{-x} = 2\sqrt{2}$. Let $e^x = y$, so that $y^2 - 2\sqrt{2}y + 1 = 0$, or $y = \sqrt{2}\pm 1$. Since we want $x \geq 0$ (the definition of sech^{-1}), we need $y \geq 1$, *i.e.*, $y = \sqrt{2}+1$ and $x = \boxed{\ln(\sqrt{2}+1)}$.

Problem 5. (a) Find $\frac{d}{dx}(\sin x)^{\ln x}$.

(b) Evaluate $\int_0^2 \left(\int_2^x e^{t^2}\,dt\right)dx$.

SOLUTION: (a) Use logarithmic differentiation:

$$\left(\ln(\sin x)^{\ln x}\right)' = \left(\ln x \ln(\sin x)\right)' = \frac{\ln(\sin x)}{x} + \ln x \cot x, \quad \text{and}$$

$$\left((\sin x)^{\ln x}\right)' = \boxed{(\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \cot x\right)}.$$

(b) Integrate by parts (keeping in mind the fundamental theorem of calculus)

$$\int_0^2\left(\int_2^x e^{t^2}\,dt\right)dx = \begin{bmatrix} u = \int_2^x e^{t^2}\,dt & du = e^{x^2}\,dx \\ v = x & dv = dx \end{bmatrix} = \left[x\int_2^x e^{t^2}\,dt\right]_0^2 - \int_0^2 xe^{x^2}\,dx = 0 - \left[\frac{1}{2}e^{x^2}\right]_0^2 = \boxed{\frac{1-e^4}{2}}.$$

(In the last integral, we use the substitution $x^2 = t$. The exintegral term is $\left[x\int_2^x\right]_0^2 = 2\int_2^2 - 0\int_2^0 = 0$.)