TRANSCENDENTAL FUNCTIONS

MATH 101

CONTENTS

1. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

By definition, let $\ln x = \int_1^x dx/x$ and $\exp x = e^x = \ln^{-1} x$ (the inverse function). If x is a rational number, then e^x does coincide with the x-th power of the number $e = \exp(1)$. Other powers are defined via $a^x = e^{x \ln a}$ (for $a > 0$). The following identities hold:

$$
\begin{aligned}
(\ln x)' &= 1/x, & (a^x)' &= a^x \ln a, & (ab)^x &= a^x b^x,\\
\ln(xy) &= \ln x + \ln y, & a^{x+y} &= a^x a^y, & (a/b)^x &= a^x/b^x.\\
\ln(x/y) &= \ln x - \ln y, & a^{x-y} &= a^x/a^y, & &\\
\ln(x^n) &= n \ln x, & a^{nx} &= (a^x)^n, & &
\end{aligned}
$$

$\ln x$ is defined for $x > 0$ and takes all real values. e^x is defined for all real x and takes all positive values. Both the functions are increasing. One has

$$
\begin{aligned}
\ln 0 &= 1, & \lim_{x\to+\infty} \ln x &= +\infty, & \lim_{x\to 0^+} \ln x &= -\infty,\\
e^0 &= 1, & \lim_{x\to+\infty} e^x &= +\infty, & \lim_{x\to-\infty} e^x &= 0.
\end{aligned}
$$

See also integrals involving exponential and hyperbolic functions and integration by parts.

2. LOGARITHMIC DIFFERENTIATION

Sometimes it is easier to find the derivative of the function $\ln f(x)$ and then use the formula $f'(x) = f(x)(\ln f(x))'$.

2.1. Example. Let $f(x) = (x+2)^5(x^2-1)^7(x+3)^9$. Then $\ln f(x) = 5\ln(x+2) + 7\ln(x^2-1) + 9\ln(x+3)$ and

$$f'(x) = (x+2)^5(x^2-1)^7(x+3)^9\Big(\frac{5}{x+2} + \frac{14x}{x^2-1} + \frac{9}{x+3}\Big).$$

Remark. Strictly speaking, the calculation is only valid when $x+2>0$, $x^2-1>0$, and $x+3>0$. However, the final result (without logarithms) holds for all values of x.

2.2. Example. Let $f(x) = x^{x^2}$. Then $\ln f(x) = x^2 \ln x$, $(\ln f(x))' = 2x\ln x + x$, and $f'(x) = x^{x^2}(2x\ln x + x)$.

Remark. Logarithmic differentiation is the **only** way to do this one. Do **not** try to use $(x^n)' = nx^{n-1}$ (which assumes $n = \text{const}$) or $(a^x)' = a^x \ln a$ (which assumes $a = \text{const}$)!

3. Inverse trigonometric functions

The important ones are $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, and $\arctan x = \tan^{-1} x$. (Do not confuse the inverse function $\sin^{-1} x$ and the reciprocal $(\sin x)^{-1} = 1/\sin x$!) Here are some properties. (I also list the inverse hyperbolic functions here; see below.)

Function	Derivative	Domain	Range	Alternative formula
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	$[-1,1]$	$[-\pi/2,\pi/2]$	
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$	$[-1,1]$	$[0,\pi]$	
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$	
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$(-\infty,\infty)$	$(-\infty,\infty)$	$\ln(x+\sqrt{x^2+1})$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$	$[1,\infty)$	$[0,\infty)$	$\ln(x+\sqrt{x^2-1})$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$	$(-1,1)$	$(-\infty,\infty)$	$\dfrac{1}{2}\ln\dfrac{1+x}{1-x}$

4. Hyperbolic functions

The functions are defined via $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x/\cosh x$. They share most of the properties of the trigonometric functions, but **beware of the signs!**

	Trigonometric identities	Hyperbolic identities
1.	$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$	$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x$
2.	$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
3.	$1 + \tan^2 x = 1/\cos^2 x$	$1 - \tanh^2 x = 1/\cosh^2 x$
4.	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5.	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6.	$\sin 2x = 2 \sin x \cos x$	$\sinh 2x = 2 \sinh x \cosh x$
7.	$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$	$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$
	$= 1 - 2\sin^2 x$	$= 1 + 2\sinh^2 x$
8.	$\sin^2 x = (1 - \cos 2x)/2$	$\sinh^2 x = (\cosh 2x - 1)/2$
9.	$\cos^2 x = (1 + \cos 2x)/2$	$\cosh^2 x = (\cosh 2x + 1)/2$
10.	$\sin x \sin y = -\frac{1}{2}(\cos(x+y) - \cos(x-y))$	$\sinh x \sinh y = \frac{1}{2}(\cosh(x+y) - \cosh(x-y))$
11.	$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$	$\cosh x \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y))$
12.	$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$	$\sinh x \cosh y = \frac{1}{2}(\sinh(x+y) + \sinh(x-y))$

4.1. Remark. For those familiar with complex numbers: all the hyperbolic identities can easily be obtained from the corresponding trigonometric ones by a formal substitution. In fact, the two sets of functions are closely related: one has

$$\sinh x = -i\sin(ix), \qquad \cosh x = \cos(ix), \qquad \tanh x = -i\tan(ix),$$
$$\sin x = -i\sinh(ix), \qquad \cos x = \cosh(ix), \qquad \tan x = -i\tanh(ix),$$

(where $i^2 = -1$. These formulas do make sense, but we will not go too deep into the details.)

Example. Consider the identity $\cos 2y = \cos^2 y - \sin^2 y$ and express the trigonometric functions in terms of their hyperbolic counterparts: $\cosh 2iy = \cosh^2 iy - (-i\sin iy)^2$. Now let $iy = x$ and notice that $i^2 = -1$; we get $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Example. Similarly, from the identity $\sin 2y = 2\sin y\cos y$ we get $-i\sinh 2x = -2i\sinh x\cosh x$, and it remains to cancel $-i$ to get the hyperbolic identity.

See also integrating trigonometric and hyperbolic expressions and trigonometric/hyperbolic substitutions.