INTEGRATION TECHNIQUES

Math 101

Contents

References: Thomas & Finney, *Calculus and Analytic Geometry,* **9th Edition**:
4.3, 4.8, 7.2, 7.3 [**advanced**], 7.4.

There is **no** general rule for evaluating an integral. Each time you have to look for a formula/rule that would simplify the integral and eventually reduce it to table ones.

1. Substitution (§§4.3, 4.8)

Use these formulas (direct and backward):

$$\int f(g(x))g'(x)\,dx = \begin{bmatrix} u = g(x) \\ du = g'(x)\,dx \end{bmatrix} = \int f(u)\,du = F(u) + C = F(g(x)) + C,$$

$$\int f(x)\,dx = \begin{bmatrix} x = g(u) \\ dx = g'(u)\,du \end{bmatrix} = \int f(g(u))g'(u)\,du = F(u) + C = F(g^{-1}(x)) + C,$$

$$\int f(kx+b)\,dx = \frac{1}{k}F(kx+b) + C, \text{ where } \int f(x)\,dx = F(x) + C \quad \text{(important special case).}$$

For definite integrals the former formula takes the form

$$\int_a^b f(g(x))g'(x)\,dx = \begin{bmatrix} u = g(x) \\ du = g'(x)\,dx \end{bmatrix} = \int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)}.$$

Do not forget that dx also needs substitution! (In an expression like $\int f(g(x))\,dx$ you **cannot** substitute $u = g(x)$!) Do not forget to return to the original variable (or recalculate limits in case of definite integrals)! Some standard substitutions will be discussed later (4, 5, 6, 7).

2. Integration by parts (§7.2)

$$\int u\,dv = uv - \int v\,du, \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Success of the method depends on the decomposition of the integrand in the form $u\,dv$. Choose for dv something that you can easily integrate (as you need $u = \int du$ for the formula)! Proceed only if the resulting integral $\int v\,du$ is simpler that the original one and you know what to do with it! (Certain integrals, e.g., $\int e^x \sin x\,dx$ below, are an exception: integration by parts may give the same integral, but the resulting expression can be treated as an equation.) Below are some typical examples.

2.1. $\int x^n \sin x\,dx$, $\int x^n \cos x\,dx$, $\int x^n e^x dx$. Take $u = x^n$. Integration by parts is to be applied n times (with $u = x^n$), each time reducing the power of x by 1.

2.2. $\int e^x \sin x\,dx$, $\int e^x \cos x\,dx$. Integrate by parts twice, **each time** taking $u = e^x$. Solve the resulting equation in the original integral.

2.3. $\int x^n (\ln x)^m dx$. Take $u = (\ln x)^m$. Integration by parts reduces m by 1. Do it m times.

3. Rational functions (§7.3) [advanced]

A rational function $P(x)/Q(x)$, where P and Q are polynomials in x, can be integrated in three steps:

(1) Reduce the fraction if it is improper (i.e., $\deg P \geqslant \deg Q$);
(2) Decompose $P(x)/Q(x)$ into a sum of partial fractions;
(3) Integrate each partial fraction.

3.1. Step 1. Reducing an improper fraction. Do not overlook this step! Otherwise, Step 2 will produce an erroneous result! If the fraction is improper, i.e., $\deg P \geqslant \deg Q$, then divide P by Q and write $P/Q = F + (R/Q)$, where F is the ratio and R is the remainder. The polynomial $F(x)$ can easily be integrated. With the proper fraction $R(x)/Q(x)$ proceed to Step 2.

3.2. Step 2. Decomposing into partial fractions. From now on assume that the fraction $P(x)/Q(x)$ is **proper** (i.e., $\deg P < \deg Q$, see above). We need to decompose the denominator $Q(x)$ into the product of linear (i.e., of the form $(x - r)$) and **irreducible** quadratic (i.e., of the form $(x^2 + px + q)$ with $p^2 - 4q < 0$) factors. Now the decomposition into partial fractions can be found by the method of undetermined coefficients. Write the identity $P(x)/Q(x) = \dots$, where the right hand side is composed as follows: each linear factor $(x - r)^m$ contributes m terms

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m},$$

and each quadratic factor $(x^2 + px + q)^n$ contributes n terms

$$\frac{B_1 x + C_1}{x^2+px+q} + \frac{B_2 x + C_2}{(x^2+px+q)^2} + \dots + \frac{B_n x + C_n}{(x^2+px+q)^n}.$$

(Of course, each coefficient in the resulting expression should be given it's own name. The total number of undetermined coefficients A, B, C, ... must be equal to the degree of Q.) In the resulting identity get rid of the denominators (by multiplying both the sides by $Q(x)$) and write down equations for the undetermined coefficients. There are two ways to obtain equations: (1) by equating the coefficients of equal powers of x (typically, simpler equations are obtained from higher powers), and (2) by substituting particular values of x (usually some simple values like $x = 0$, ± 1, etc. or the values $x = r$, where $(x - r)$ is a factor in the decomposition of $Q(x)$).

> **Important Remark:** The resulting system of linear equations in $A, B, C, \cdots$ must have a **unique solution!** If there is no solutions, something is wrong in your calculation. If there are many solutions, then either something is wrong or there is not enough equations.

3.3. Step 3. Integrating partial fractions. Here are the formulas (with C omitted):

$$\int \frac{dx}{x-r} = \ln|x-r|, \qquad \int \frac{dx}{(x-r)^n} = \frac{1}{(1-n)(x-r)^{n-1}} \quad \text{(substitution } x - r = t\text{)}.$$

For the other partial fractions, first complete the square:

$$\int \frac{Bx + C}{(x^2+px+q)^n}\,dx = \int \frac{Bt + C'}{(t^2+a^2)^n}\,dt \quad \text{(where } t = x + \frac{p}{2},\ a^2 = \frac{4q-p^2}{4} > 0, \text{ and } C' = C - \frac{Bp}{2}\text{)}.$$

Then

$$\int \frac{t\,dt}{t^2+a^2} = \frac{1}{2}\ln(t^2+a^2), \quad \int \frac{t\,dt}{(t^2+a^2)^n} = \frac{1}{2(1-n)(t^2+a^2)^{n-1}} \quad \text{(substitution } t^2+a^2=u\text{)},$$

$\int \frac{dt}{t^2+a^2} = \frac{1}{a}\tan^{-1}\frac{t}{a}$ is a table integral, and the remaining integral $I_n = \int \frac{dt}{(t^2+a^2)^n}$ is evaluated in n steps via integration by parts and reducing n:

$$\begin{aligned} I_n &= \frac{1}{a^2}\int \frac{t^2+a^2-t^2}{(t^2+a^2)^n}dt = \frac{1}{a^2}I_{n-1} - \frac{1}{2(n-1)a^2}\int t\cdot d\Big(\frac{1}{(t^2+a^2)^{n-1}}\Big) \\ &= [\text{by parts}] = \frac{1}{a^2}I_{n-1} - \frac{1}{2(n-1)a^2}\frac{t}{(t^2+a^2)^{n-1}} + \frac{1}{2(n-1)a^2}I_{n-1}. \end{aligned}$$

4. Trigonometric functions (§7.4, including exercises)

An integral of a rational function of $\sin x$ and $\cos x$ can always be reduced to integrating a rational function. The best thing to try is using trigonometric identities (see `transc.pdf`) to convert products to sums and to reduce powers. Below are a few standard hints.

4.1. The universal trigonometric substitution [advanced]**.** One lets $t = \tan(x/2)$. Then $x = 2\tan^{-1} t$ and

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}$$

are expressed in terms of t rationally.

> **Important Remark:** This substitution is universal. However, typically it leads to a huge amount of calculations. Thus, use it only if you cannot think of a better approach!
>
> **Important Remark:** Always try to simplify the expression using trigonometric identities. The general rule is the following: the bigger the argument of the functions (sin and cos), the lower the degree of the expression. However, keep in mind that in general you want to have all functions of the same argument! Below are a few examples.

4.2. Integrals of the form $\int R(\sin^2 x, \cos x)\sin x\,dx$ and $\int R(\sin x, \cos^2 x)\cos x\,dx$, where R is a rational function. Use the substitution $\cos x = t$, $\sin^2 x = 1-t^2$, $\sin x\,dx = -dt$ (in the former case) or $\sin x = t$, $\cos^2 x = 1-t^2$, $\cos x\,dx = dt$ (in the latter case).

Example.

$$\int \sin x\cos^3 x\,dx = \int \sin x(1-\sin^2 x)\cos x\,dx = \int (u-u^3)du = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C$$

(where $u = \sin x$ and $du = \cos x\,dx$).

4.3. Integrals of the form $\int R(\sin x, \cos x)\,dx$, where R is a rational function whose **all terms have even degree** In other words, the integrand can be expressed in terms of $\sin^2 x$, $\cos^2 x$, and $\sin x\cos x$. Reduce the degree using the formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin x\cos x = \frac{1}{2}\sin 2x \quad \text{(see transc.pdf)}.$$

Example.

$$\int \sin x\cos^3 x\,dx = \int \frac{1}{2}\sin 2x\frac{1+\cos 2x}{2}dx = \int \Big(\frac{\sin 2x}{4} + \frac{\sin 4x}{8}\Big)dx = -\frac{\cos 2x}{8} - \frac{\cos 4x}{16} + C.$$

4.4. Integrals $\int \tan^n x\,dx$. One has

$$\int \tan x\,dx = -\ln|\cos x|, \quad \int \tan^n x\,dx = \int \tan^{n-2} x\Big(1 - \frac{1}{\cos^2 x}\Big)dx = \int \tan^{n-2} x\,dx - \frac{\tan^{n-1} x}{n-1}.$$

4.5. Integrals $\int \sin mx\sin nx\,dx$, $\int \sin mx\cos nx\,dx$, $\int \cos mx\cos nx\,dx$. Convert products of functions to sums (see trigonometric identities in `transc.pdf`).

5. Exponential and hyperbolic functions

An integral of the form $\int R(e^x)\,dx$ (where R is a rational function) can be reduced to integrating a rational function by the substitution $e^x = t$, $x = \ln t$, $dx = dt/t$. An integral of the form $\int R(\sinh x, \cosh x)\,dx$ can be treated either by expressing $\sinh x$ and $\cosh x$ in terms of e^x or, better yet, similar to 4, using corresponding hyperbolic identities (see `transc.pdf`).

6. Trigonometric and hyperbolic substitutions (§7.4)

Assume that we need to integrate an expression rational in x and the radical $\sqrt{ax^2+bx+c}$. First, complete the square

$$ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right), \quad \text{where, as usual, } D = b^2-4ac,$$

and make the substitution $u = x + \frac{b}{2a}$, $du = dx$. Now, depending on the signs of a and D, we need to treat one of the following three irrationalities: $\sqrt{u^2+r^2}$, $\sqrt{u^2-r^2}$, or $\sqrt{r^2-u^2}$ (where $r = \sqrt{|D/4a^2|}$).

6.1. Example. $\sqrt{3+6x-3x^2} = \sqrt{-3(x^2-2x-1)} = \sqrt{-3((x-1)^2-2)} = \sqrt{3}\sqrt{2-u^2}$. Here $r = \sqrt{2}$.

Now we make an appropriate trigonometric or hyperbolic substitution and reduce the given integral to a trigonometric or hyperbolic one, which can be treated as in 4 or 5, respectively.

Irrationality	Trigonometric substitution	Hyperbolic substitution
$\sqrt{r^2-u^2}$	$u = r\sin t$, $\quad -\pi/2 \leqslant t \leqslant \pi/2$ $du = r\cos t\,dt$, $\quad \sqrt{r^2-u^2} = r\cos t$ $t = \sin^{-1}(u/r)$	–
$\sqrt{u^2+r^2}$	$u = r\tan t$, $\quad -\pi/2 < t < \pi/2$ $du = r\,dt/\cos^2 t$, $\quad \sqrt{u^2+r^2} = r/\cos t$ $t = \tan^{-1}(u/r)$	$u = r\sinh t$, $\quad t$ any number $du = r\cosh t\,dt$, $\quad \sqrt{u^2+r^2} = r\cosh t$ $t = \sinh^{-1}(u/r)$
$\sqrt{u^2-r^2}$	$u = r/\cos t$, $\quad 0 \leqslant t < \pi/2$ or $\pi/2 < t \leqslant \pi$ $du = r\sin t\,dt/\cos^2 t$, $\quad \sqrt{u^2-r^2} = \mp r\tan t$ $t = \cos^{-1}(r/u)$	$u = \pm r\cosh t$, $\quad t \geqslant 0$ $du = \pm r\sinh t\,dt$, $\quad \sqrt{u^2-r^2} = r\sinh t$ $t = \cosh^{-1}(\pm u/r)$

Important Remark: To my opinion, in the last two cases the hyperbolic substitutions are usually simpler.

Important Remark: In the last case one should be very careful about the signs. The integrand is defined on two disjoint intervals, $x \geqslant 1$ and $x \leqslant -1$, which should be treated differently. (The upper sign corresponds to the first interval.) One should keep in mind that $\cosh t \geqslant 1$ for any t and that, by definition, $\sqrt{u^2-r^2} \geqslant 0$ for any u.

7. Other irrationalities

An expression rational in x and $\sqrt[m]{\dfrac{ax+b}{cx+d}}$ is integrated by the substitution $\dfrac{ax+b}{cx+d} = t^m$. Then $x = -\dfrac{dt^m-b}{ct^m-a}$ and the integrand becomes rational (see 3).

8. Table integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq 1), \qquad \int \frac{dx}{x} = \ln x + C,$$

$$\int e^x dx = e^x + C, \qquad \int a^x dx = \frac{a^x}{\ln a} + C,$$

$$\int \sin x\, dx = -\cos x + C, \qquad \int \cos x\, dx = \sin x + C,$$

$$\int \sec^2 x\, dx = \tan x + C, \qquad \int \csc^2 x\, dx = -\cot x + C,$$

$$\int \sec x \tan x\, dx = \sec x + C, \qquad \int \csc x \cot x\, dx = -\csc x + C,$$

$$\int \sinh x\, dx = \cosh x + C, \qquad \int \cosh x\, dx = \sinh x + C,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \quad (a > 0), \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \quad (a > 0),$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C \quad (a > 0),$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C \quad (a > 0), \qquad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C \quad (a > 0),$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C \quad (a > 0).$$

(see `transc.pdf` for the expressions for $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in terms of ln.)

[**advanced**] This topic has been omitted or moved to Math 102