Final Topics[1]

1. Limits of sequences. A *sequence* is a function whose domain is the set of positive integers (sometimes starting from a certain integer n_0). The usual notation for the n-th member of a sequence a is a_n.

Sequences are usually given by a formula ($a_n = f(n)$) or recursively, i.e., with few first terms given explicitly and the others defined in terms of the previous ones.

Limits of sequences are defined in the same manner as limits of functions. Working with sequences, we always mean the limit when $n \to \infty$. If a sequence has limit, it is said to *converge*; otherwise it *diverges*. Limits of sequences obey the same rules as limits of functions: there are *Sum Rule*, *Difference Rule*, *Product Rule*, *Quotient Rule*, and *Sandwich Theorem*. Furthermore, if $\lim_{n\to\infty} a_n = L$ and $f(x)$ is a function continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. (Roughly speaking, $\lim f = f(\lim)$.)

Sequences given by functions. The most useful rule for finding limits of sequences is the following: if a sequence is given by a formula $a_n = f(n)$, where $f(x)$ is a function of real argument, **and** the limit $\lim_{x\to\infty} f(x) = L$ exists, then $\lim_{n\to\infty} a_n = L$. Thus, to find the limit of most sequences you can use the standard techniques of limits of functions.

Some useful limits. Here are some limits worth remembering (x in the formulas is a constant):

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0, \qquad \lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0), \qquad \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x,$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1, \qquad \lim_{n\to\infty} x^n = 0 \quad (|x| < 1), \qquad \lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

2. Infinite series. A series is an 'infinite sum', i.e., expression of the form $\sum_{n=0}^{\infty} a_n$, where a_n is a given sequence (the sequence of *n-th terms* of the series). The series is said to *converge* (*diverge*) if its sequence of *partial sums* $S_n = \sum_{k=0}^{n} a_k$ converges (respectively, diverges). In the former case the limit $\lim_{n\to\infty} S_n$ is called the *sum* of the series.

Useful example. The *geometric series* $\sum_{n=0}^{\infty} q^n$ converges if $|q| < 1$ and diverges otherwise. In the former case the sum is $1/(1-q)$.

Main properties. Convergent series can be added, subtracted, and multiplied by constants. More precisely, if $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ converge, then so do $\sum_{n=0}^{\infty}(a_n \pm b_n) = A \pm B$ and $\sum_{n=0}^{\infty} ca_n = cA$ (where c is a constant).

Important Remark: In general, one can **not** multiply convergent series, nor can one interchange the members of a series.

Important Remark: The convergence of a series depends only on its 'behavior at infinity', i.e., series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=N}^{\infty} a_n$ converge or diverge simultaneously. (Notice the similarity with improper integrals.) Thus, in all the tests below one can assume that the hypotheses are satisfied for all 'sufficiently large' values of n.

A necessary (but far not sufficient!) condition for convergence. If a series $\sum_{n=0}^{\infty} a_n$ converges, then its n-th term tends to zero, $\lim_{n\to\infty} a_n = 0$.

Important Remark: There are lots of diverging series with $\lim_{n\to\infty} a_n = 0$!

In general, it is very difficult to find the sum of a series. Thus, we will mainly be concerned with their convergence.

3. Series with nonnegative terms. The tests listed here apply **only** if all the terms of the series are nonnegative.

The comparison test. Assume that $0 \leqslant a_n \leqslant b_n$ for all sufficiently large n. Then:

(1) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$;
(2) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.

The limit comparison test. Assume that $a_n, b_n > 0$ for all sufficiently large n.

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ and $\sum_{n=0}^{\infty} b_n$ converges, then so does $\sum_{n=0}^{\infty} a_n$.

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ and $\sum_{n=0}^{\infty} b_n$ diverges, then so does $\sum_{n=0}^{\infty} a_n$.

[1] See also Midterm I Topics, Midterm II Topics, and Integration

These series are useful for comparison purposes:

(1) the *geometric series* $\sum_{n=0}^{\infty} q^n$ converges if and only if $|q| < 1$.
(2) the *generalized harmonic series* $\sum_{n=0}^{\infty}(1/n^p)$ converges if and only if $p > 1$.

Next is a very powerful test that solves the convergence question for most reasonable series.

The integral test. Assume that $a_n = f(n)$ for some positive, continuous, **decreasing** function $f(x)$. Then the series $\sum_{n=0}^{\infty} a_n$ and the integral $\int_a^{\infty} f(x)\,dx$ converge or diverge simultaneously. (Here a is any number.)

Finally, here are two intrinsic tests that do not use any other series or function explicitly.

The ratio test (*D'Alambert criterion*). Let $\sum_{n=0}^{\infty} a_n$ be a series with positive terms, and suppose that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \rho$. Then:

(1) if $\rho < 1$, the series converges;
(2) if $\rho > 1$, the series diverges.

The n-th root test (*Cauchy criterion*). Let $\sum_{n=0}^{\infty} a_n$ be a series with positive terms, and suppose that $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then:

(1) if $\rho < 1$, the series converges;
(2) if $\rho > 1$, the series diverges.

Important Remark: Both D'Alambert and Cauchy criteria leave the question open if $\rho = 1$.

4. Arbitrary series: absolute and conditional convergence. A series $\sum_{n=0}^{\infty} a_n$ (with arbitrary terms) is said to *converge absolutely* if the series $\sum_{n=0}^{\infty}|a_n|$ converges. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty}|a_n|$ does not, the series is said to converge *conditionally.*

Absolute convergence implies convergence. If $\sum_{n=0}^{\infty}|a_n|$ converges, then so does $\sum_{n=0}^{\infty} a_n$.

Important Remark: Thus, in many cases one can conclude that a series converges by applying one of the above tests to the series $\sum_{n=0}^{\infty}|a_n|$ with nonnegative terms. However, if $\sum_{n=0}^{\infty}|a_n|$ diverges, one can **not** conclude that $\sum_{n=0}^{\infty} a_n$ diverges.

If the absolute convergence fails, one can try to conclude that the original series still converges (conditionally) using the following Lagrange theorem. (Not that the theorem applies to **very** special series, but most reasonable series with arbitrary terms **are** of this kind!)

Lagrange theorem. Assume that the series has the form $\sum_{n=0}^{\infty}(-1)^n c_n$, where all c_n are nonnegative, the sequence c_n is decreasing, and $\lim_{n\to\infty} c_n = 0$. Then the series converges.

Addition to the Lagrange theorem. In fact, the Lagrange theorem tells us more: under the hypotheses the error of replacing the sum S of the series by its partial sum $S_n = \sum_{k=0}^{n}(-1)^k c_k$ does not exceed the first dropped term c_{n+1}: $|S - S_n| \leqslant c_{n+1}$.

5. Power series. A *power series* is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$. The c_n's are called the *coefficients* of the series and a its *center.* The sum of a power series and even its convergence depend on x.

Interval of convergence. For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there is a number R (called *radius of convergence*), $0 \leqslant R \leqslant \infty$, such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. The interval $(a-R, a+R) = \{x\colon |x-a| < R\}$ is called the *interval of convergence.* (In the special case $R = 0$ the interval of convergence is a single point $x = a$; in the case $R = \infty$ it is the whole real line.)

Radius of convergence. The radius of convergence can usually be found using D'Alambert or Cauchy criterion. More precisely, if at least one of the limits $\lim\limits_{n\to\infty} \dfrac{|c_{n+1}|}{|c_n|} = L$ or $\lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = L$ exists (or is ∞), then $R = 1/L$ (with the usual convention $1/0 = \infty$ and $1/\infty = 0$).

The endpoints. The behaviour (absolute/conditional convergence) of a power series at the endpoints $x = a - R$ and $x = a + R$ of its interval of convergence should **always** be investigated separately (unless $R = 0$ or ∞) by substituting the particular value of x and investigating the resulting numeric series.

Within the interval of convergence power series can be integrated, differentiated, and multiplied.

Differentiation. If a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges to a function $f(x)$ for $|x-a| < R$, then the series $\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$, obtained by differentiating the terms of the original series, converges to the derivative $f'(x)$ for $|x-a| < R$.

Differentiation. If a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges to a function $f(x)$ for $|x-a|<R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^{n+1}/(n+1)$, obtained by integrating the terms of the original series, converges to the integral $\int_a^x f(t)\,dt$ for $|x-a|<R$.

Multiplication. If two power series $\sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$ and $\sum_{n=0}^{\infty} d_n(x-a)^n = d(x)$ converges for $|x-a|<R$, then the series $\sum_{n=0}^{\infty}\bigl(\sum_{k=0}^{n} c_x d_{n-k}\bigr)(x-a)^n$, obtained by the multiplication of the two original series, converges to the product $f(x)g(x)$ for $|x-a|<R$.

6. Maclaurin and Taylor series. The principal usage of power series is representing functions. Given a function $f(x)$ and a number a so that f has derivatives of all orders at a, one can write down the *Tailor series* of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{(where } f^{(0)}(a)=f(a),\ 0!=1, \text{ and } (x-a)^0=1).$$

(In the special case $a=0$ it is called *Maclaurin series.*) This is the only series (centered at a) that **may** converge to $f(x)$. However, one still has to answer two questions:

- Where does the series converge? (What is its interval of convergence?)
- If the series does converge, is its sum equal to $f(x)$?

The answer is usually given by the following *Taylor formula with remainder*:

$$f(x)=\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(c_N(x))}{(N+1)!}(x-a)^{N+1},$$

where the (unknown) number $c_N(x)$ depends on both N and x and lies between a and x. The series converges to $f(x)$ for all values of x (and only for these values) for which

$$\lim_{N\to\infty} \frac{f^{(N+1)}(c_N(x))}{(N+1)!}(x-a)^{N+1}=0.$$

To show that the limit is 0 one usually tries to estimate $|f^{(N+1)}(c_N(x))|$ from above by a simpler expression (without c) that one can handle.

Functions that can be represented by power series are called *analytic*.

Here are some Maclaurin series worth remembering. (Lots of others can be obtained from them by substitution, multiplication, integration, and differentiation.)

$$\frac{1}{(1-x)}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots, \qquad \text{(geometric series, } -1<x<1),$$

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots, \qquad (-1<x\leqslant 1),$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots, \qquad (-1<x\leqslant 1),$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots, \qquad (x \text{ any number}),$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots, \qquad (x \text{ any number}),$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots, \qquad (x \text{ any number}),$$

$$(1+x)^m=1+\sum_{n=1}^{\infty}\binom{m}{n}x^n, \quad \text{where } \binom{m}{n}=\frac{m(m-1)\dots(m-n+1)}{n!}$$

$$\text{(binomial series, } -1<x<1).$$

If m is a positive integer, the last series terminates at $n=m$ and becomes the usual binomial formula.

7. Applications of power series.

Numeric calculation. The value $f(x)$ can be evaluated by taking a few first terms of the power series for $f(x)$. The remainder formula above gives one a good way to estimate the truncation error. For example, the series for e^x, $\sin x$, and $\cos x$ converge very fast (for small values of x) and good results can be obtained by taking very few terms.

Evaluating nonelementary integrals. One can evaluate integrals by representing the integrand by a power series and integrating termwise. For example,

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \qquad \text{for any } x.$$

Euler's formula. By substituting $x = iy$ to the series for e^x and rearranging the terms one obtains $e^{iy} = \cos y + i \sin y$. An advanced study shows that this is the only reasonable way to define e^{iy} with retaining all the nice properties (like differentiability) of this function.