Solutions to Midterm I

Problem 1. Find the limits (without using l'Hfpital's rule):

(a) $\lim_{x\to 0}\dfrac{\sqrt{\cos 3x}-1}{x\sin 2x}$.

(b) $\lim_{x\to -2}\dfrac{x^2+x-2}{x^3+2x^2-2x-4}$.

SOLUTION:

$$(a)\qquad \lim_{x\to 0}\frac{\sqrt{\cos 3x}-1}{x\sin 2x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos 3x-1}{x\sin 2x(\sqrt{\cos 3x}+1)}=-\lim_{x\to 0}\frac{2\sin^2(3x/2)}{x\sin 2x(\sqrt{\cos 3x}+1)}=$$
$$-\frac{3}{2}\cdot\frac{3}{2}\cdot\lim_{x\to 0}\frac{\sin(3x/2)}{3x/2}\cdot\lim_{x\to 0}\frac{\sin(3x/2)}{3x/2}\cdot\lim_{x\to 0}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0}\frac{1}{\sqrt{\cos 3x}+1}=\boxed{-\frac{9}{8}}.$$

$$(b)\qquad \lim_{x\to -2}\frac{x^2+x-2}{x^3+2x^2-2x-4}=\left[\frac{0}{0}\right]=\lim_{x\to -2}\frac{(x+2)(x-1)}{(x+2)(x^2-2)}=\lim_{x\to -2}\frac{x-1}{x^2-2}=\boxed{-\frac{3}{2}}.$$

(In (b), once we know that the polynomials vanish at $x=-2$, we factor them by simple division.)

Problem 2. Find the derivatives of the following functions:

(a) $y=4\sqrt{1+\sin\sqrt{x}}$.

(b) $y=\tan^4(\pi x^3)-\tan^4(\pi^4)$.

(c) $y=\left(\dfrac{1+\sin x}{1-\cos x}\right)^2$.

SOLUTION:

$$(a)\qquad \left(4\sqrt{1+\sin\sqrt{x}}\right)'=\frac{4\big(1+\sin\sqrt{x}\big)'}{2\sqrt{1+\sin\sqrt{x}}}=\frac{2\cos\sqrt{x}\cdot\big(\sqrt{x}\big)'}{\sqrt{1+\sin\sqrt{x}}}=\boxed{\frac{\cos\sqrt{x}}{\sqrt{x}\sqrt{1+\sin\sqrt{x}}}}.$$

$$(b)\qquad \big(\tan^4(\pi x^3)-\tan^4(\pi^4)\big)'=4\tan^3(\pi x^3)\cdot\big(\tan(\pi x^3)\big)'=\frac{4\tan^3(\pi x^3)}{\cos^2(\pi x^3)}\cdot(\pi x^3)'=\boxed{\frac{12\pi x^2\tan^3(\pi x^3)}{\cos^2(\pi x^3)}}.$$

$$(c)\qquad \left[\left(\frac{1+\sin x}{1-\cos x}\right)^2\right]'=2\cdot\frac{1+\sin x}{1-\cos x}\cdot\left(\frac{1+\sin x}{1-\cos x}\right)'=$$
$$\frac{2(1+\sin x)\big((1+\sin x)'(1-\cos x)-(1+\sin x)(1-\cos x)'\big)}{(1-\cos x)^3}=$$
$$\frac{2(1+\sin x)(\cos x-\cos^2 x-\sin x-\sin^2 x)}{(1-\cos x)^3}=\boxed{\frac{2(1+\sin x)(\cos x-\sin x-1)}{(1-\cos x)^3}}.$$

Problem 3. Find the tangents to the parabola $y=-x^2$ passing through the point $(1,3)$.

SOLUTION: The tangent to the parabola at a point (x_0,y_0) (where, of course, $y_0=-x_0^2$) has the form $(Y-y_0)=f'(x_0)(X-x_0)$, *i.e.*, in our case, $Y+x_0^2=-2x_0(X-x_0)$. This line should pass through the point $X=1$, $Y=3$. Plug it in to get $3+x_0^2=-2x_0(1-x_0)$ and solve this equation for x_0. We get $x_0^2-2x_0-3=0$ and $x_0=-1$ or $x_0=3$. The corresponding tangents are found from the above equation: $\boxed{Y=2X+1}$ (for $x_0=-1$) and $\boxed{Y=-6X+9}$ (for $x_0=3$).

Problem 4. Find y'' at the point $(x,y)=(0,\pi)$, where y is a differentiable function of x satisfying the equation $\sin y=x^2\cos^2 y$.

SOLUTION: Differentiate the given equation: $\cos y \cdot y' = 2x\cos^2 y + x^2 \cdot 2\cos y \cdot (-\sin y) \cdot y'$. Collecting the terms with y' and cancelling $\cos y$ gives $y'(1 + 2x^2 \sin y) = 2x \cos y$. Call this $(*)$. To find y'', we need to differentiate it once more. Notice that from $(*)$ it follows that, at the point in question, one has $y' = 0$. Thus, when differentiating $(*)$, we can disregard all terms (in the result!) containing y', *i.e.,* there is no need to write them down: $y''(1 + 2x^2 \sin y) + y'(\ldots) = 2\cos y + (\ldots)y'$. Now plug in $x = 0$, $y = \pi$, and $y' = 0$ to get $\boxed{y'' = -2}$.

Problem 5. A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

SOLUTION: As usual (the Galileo principle), we can assume that the plane is standing still, find the speed of the car, and add 120 to the result.

Let $h = 3\,\text{mi}$ be the (constant) plane altitude, $x = x(t)$, the distance between the car and the projection of the plane to the highway, and $D = D(t)$, the line-of-sight distance. Then the given data amount to $D = 5$ and $dD/dt = -160$ (the distance is **decreasing**), and we need to find dx/dt. The Pythagorean theorem gives us a relation: $x^2 + h^2 = D^2$. Differentiate it to get $2x(dx/dt) = 2D(dD/dt)$. Here, D and dD/dt are given, and $x = \sqrt{D^2 - h^2}$ is found from the original relation. Thus, $dx/dt = D(dD/dt)/\sqrt{D^2 - h^2}$. Now use the numeric data ($dx/dt = 5 \cdot (-160)/4 = -200$), do not forget to add 120, and take absolute value (as the police would not accept negative speed in the report). Finally, $\boxed{\text{car's speed is } 80\,\text{mi/h}}$. Extra credit question: how much will this cost to the driver?