Solutions to Midterm I

Problem 1. Find the limits (without using l'Hôpital's rule):

(a) $\lim\limits_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

(b) $\lim\limits_{x\to\infty} \dfrac{2x^2+3}{5x^2+7}$

(c) $\lim\limits_{x\to\infty} \dfrac{f(5x)-f(3x)}{3x}$ assuming that $\lim\limits_{x\to\infty} f'(x)=5$.

SOLUTION:

(a) $\lim\limits_{x\to -5} \dfrac{x^2+3x-10}{x+5} = \left[\dfrac{0}{0}\right] = \lim\limits_{x\to -5} \dfrac{(x+5)(x-2)}{x+5} = \lim\limits_{x\to -5} x-2 = \boxed{-7}$.

(b) $\lim\limits_{x\to\infty} \dfrac{2x^2+3}{5x^2+7} = \left[\dfrac{\infty}{\infty}\right] = \lim\limits_{x\to\infty} \dfrac{2+(3/x^2)}{5+(7/x^2)} = \boxed{\dfrac{2}{5}}$.

(c) $\lim\limits_{x\to\infty} \dfrac{f(5x)-f(3x)}{3x} = \lim\limits_{x\to\infty} \dfrac{f'(c)(5x-3x)}{3x} = \lim\limits_{x\to\infty} \dfrac{2f'(c)}{3} = \boxed{\dfrac{10}{3}}$, where $c\in(3x,5x)$ is given by the Mean Value Theorem. Since $x\to\infty$, so does c.

Problem 2. (a) Find y', where $y=\left(\dfrac{\sin x}{1+\cos x}\right)^2$.

(b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, where $x^{2/3}+y^{2/3}=1$.

SOLUTION:

(a) Use the chain rule and the fraction rule:

$$\left(\left(\frac{\sin x}{1+\cos x}\right)^2\right)' = 2\frac{\sin x}{1+\cos x}\left(\frac{\sin x}{1+\cos x}\right)'$$

$$= 2\frac{\sin x}{1+\cos x}\,\frac{\cos x(1+\cos x)-\sin x(-\sin x)}{(1+\cos x)^2} = 2\frac{\sin x(\cos x+1)}{(1+\cos x)^3} = \boxed{\frac{2\sin x}{(1+\cos x)^2}}.$$

(b) Differentiate the equation $x^{2/3}+y^{2/3}=1$ to get $x^{-1/3}+y^{-1/3}y'=0$. Thus, $\boxed{y'=-(x/y)^{-1/3}}$. Now differentiate this expression to find y'':

$$y'' = \frac{1}{3}\left(\frac{x}{y}\right)^{-4/3}\frac{y-xy'}{y^2} = \frac{1}{3}\left(\frac{x}{y}\right)^{-4/3}\frac{y+x(x/y)^{-1/3}}{y^2} = \frac{1}{3}\left(\frac{x}{y}\right)^{-4/3}\frac{y^{2/3}+x^{2/3}}{y^{5/3}} = \boxed{\frac{1}{3x^{4/3}y^{1/3}}}.$$

Problem 3. Find the points on the curve $x^2+xy+y^2=7$ where the tangent is parallel to the x-axis. Write the equations of the tangents at these points.

SOLUTION: First, differentiate the expression $x^2+xy+y^2=7$ to find y':

$$2x+y+xy'+2yy'=0, \quad \text{hence} \quad y'=-\frac{2x+y}{x+2y}.$$

The tangent is parallel to the x-axis if and only if $y'=0$, i.e., $2x+y=0$ or $y=-2x$. From the original equation it follows then that $x^2-2x^2+4x^2=7$, i.e., $x=\pm\sqrt{7/3}$. Thus, the points in question are $\boxed{(x_0,y_0)=(\pm\sqrt{7/3},\mp 2\sqrt{7/3})}$ and the tangents are found using the formula $y-y_0=y'(x_0)(x-x_0)$ (and the fact that $y'(x_0)=0$): $\boxed{y=\mp 2\sqrt{7/3}}$.

Problem 4. A right triangle whose hypotenuse is $\sqrt{3}$ meters long is revolved around one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

SOLUTION: Let r, h, and V be the radius, height, and volume, respectively. Denote by $l=\sqrt{3}$ the hypotenuse. Then $V=\pi r^2h/3$. By the Pythagorean theorem one has $r^2+h^2=l^2$. Hence, $r^2=l^2-h^2$ and $V=\pi h(l^2-h^2)/3$. This function is to be maximized on the interval $(0,l)$, which can be replaced with the segment $[0,l]$.

Find the critical points: $V'=\pi l^2/(3-h^2)=0$ yields $h=l/\sqrt{3}$. Comparing the values $V(0)=0$, $V(l)=0$, and $V(l/\sqrt{3})=2\pi l^3/9\sqrt{3}>0$ one concludes that the maximal volume of the cone $V=\boxed{2\pi l^3/9\sqrt{3}=2\pi/3}$ is attained when $h=\boxed{l/\sqrt{3}=1}$ and $r=\sqrt{l^2-h^2}=\boxed{l\sqrt{2/3}=\sqrt{2}}$.

Problem 5. Sketch the graph of the function

$$f(x) = \frac{x-1}{x^2(x-2)}$$

by finding the symmetry (if any), dominant terms, asymptotes, intervals of increasing and decreasing, extreme points, concavity, and points of inflexion.

SOLUTION: There is no symmetry. The dominant term at infinity is $1/x^2$. Furthermore, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote. The vertical asymptotes correspond to the roots of the denominator: $x = 0$ and $x = 2$.

The first derivative $y' = \dfrac{x^2(x-2) - (x-1)(2x)(x-2) - (x-1)x^2}{x^4(x-2)^2} = -\dfrac{2x^2-5x+4}{x^3(x-2)^2}$ has no roots (the numerator is always positive) and is defined whenever the original function is. Hence, there is no critical points. The second derivative is

$$-\frac{(4x-5)x^3(x-2)^2 - (2x^2-5x+4)(3x^2)(x-2)^2 - (x^2-2x+2)x^3(2)(x-2)}{x^6(x-2)^4} = 2\frac{3x^3-12x^2+20x-12}{x^4(x-2)^3}.$$

Well ... The equation $y'' = 0$ does have a single real root, but it is not easy to find it. So, we will not investigate the concavity of the graph.

It remains to determine the sign of y' (i.e., the intervals of increasing/decreasing of the function):

x		0		2	
y'	$+$	∞	$-$	∞	$-$
y	$\nearrow$	∞	$\searrow$	∞	$\searrow$

Here is the graph: