INTEGRALS

Math 101

Contents

1. Anti-derivatives and indefinite integrals

A function $F(x)$ is called an *anti-derivative* of $f(x)$ if $F'(x) = f(x)$. Any continuous function has anti-derivatives. Any two anti-derivatives of a given function differ by a constant. The set of all anti-derivatives of $f(x)$ is called the *indefinite integral* of $f(x)$; the typical notation is $\int f(x)\,dx = F(x) + C$, where $F(x)$ is one of the anti-derivatives.

> **Important Remark:** As in the case of derivatives, the variable of integration (indicated by the dx pattern) is **very** important! In an expression like $\int (ax + x^2)da$ the letter x should be treated as a constant; thus, $\int (ax+x^2)da = \frac{1}{2}a^2x + x^2a + C$, whereas $\int (ax+x^2)dx = \frac{1}{2}ax^2 + \frac{1}{3}x^3 + C$.

2. Riemann sums, definite integrals

The expression $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called the *Riemann sum* of $f(x)$. (Here $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a *partition* of a segment $[a,b]$, $c_i \in [x_{i-1}, x_i]$ are some points, and $\Delta x_i = x_i - x_{i-1}$.) The limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called the *definite integral* of $f(x)$ from a to b and is denoted by $\int_a^b f(x)\,dx$. The variable of integration (= *dummy variable*) is important (see the remark in 1), although it disappears in the result: the result of evaluation of a definite integral in respect to x must **not** contain x! For example,

$$\int_0^1 (ax+x^2)da = \frac{1}{2}x + x^2 \quad (\text{no } a), \qquad \int_0^1 (ax+x^2)dx = \frac{1}{2}a + \frac{1}{3} \quad (\text{no } x).$$

If f is continuous on $[a, b]$, the integral exists. (See also 8 for *improper integrals*.) Geometrically, the integral is the (signed) area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

2.1. Main properties:.

(1) $\int_a^a f(x)\,dx = 0$ (definition);
(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (definition);
(3) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$;
(4) $\int_a^b c_1 f_1(x) + c_2 f_2(x))dx = c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx$, where $c_1, c_2 = \text{const}$;
(5) if $a \leqslant b$ and $f(x) \geqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$;
(6) if $a \leqslant b$ and $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$, then $m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$;
(7) if f is continuous on $[a, b]$, then there is a point $c \in [a, b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$ (*the mean value theorem*).

3. The fundamental theorem of integral calculus

If $f(x)$ is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

i.e., the integral with variable upper limit is an anti-derivative of f.

3.1. Corollary (differentiation of an integral with variable limits).

$$\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(t)\,dt = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x).$$

This formula is a direct consequence of the fundamental theorem, chain rule, and property 2(2).

Example. Given that $x = \displaystyle\int_0^y \frac{dt}{(t^2+1)}$, find $\dfrac{dy}{dx}$. We use the formula above and implicit differentiation to get $1 = y'/(y^2+1)$; hence, $y' = y^2 + 1$.

3.2. Corollary (the Newton-Leibniz formula; calculation of definite integrals).

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b, \quad \text{where } F \text{ is any anti-derivative of } f.$$

This is our **principal tool** for evaluating definite integrals.

Important Remark: In applications one often needs to find an integral of the form $\int_a^b |f(x)|\,dx$, whereas it is not easy to find an anti-derivative of $|f(x)|$ (i.e., function involving absolute value). In this case one subdivides $[a, b]$ into smaller intervals $[a_i, b_i]$ (by the roots of f) so that f keeps sign within each $[a_i, b_i]$ and finds the integral as the sum of $\int_{a_i}^{b_i} \pm f(x)\,dx$.

4. Applications of definite integrals

Assume that we want to calculate a quantity S given by the naïve law $S = Ax$ (which holds whenever A does not depend on x). If A does depend on x, $A = A(x)$, then we proceed as follows: divide the interval $[a, b]$ where x changes into small subintervals. Within each small subinterval Δx_i one can assume that A does not change much and, hence, $\Delta S_i = A(x_i)\Delta x_i$. Passing to the limit, one gets $S = \int_a^b A(x)\,dx$. More precisely, for the integral formula to hold the error in the approximate formula $\Delta S_i \approx A(x_i)\Delta x_i$ must be 'much smaller' than Δx_i (i.e., of order $(\Delta x_i)^2$ or higher).

Below are some particular formulas.

5. Geometric applications

5.1. Area of a plane region. Subdivide the region to represent it as the union/difference of simple regions, each bounded by the graphs of two functions $f(x) \geqslant g(x)$ and two vertical lines $x = a$ and $x = b$. The area of one such simple region is $\int_a^b (f(x) - g(x))dx$.

Sometimes it is easier to use regions bounded by two curves $x = f(y)$ and $x = g(y)$, $f(y) \geqslant g(y)$, and two horizontal lines $y = a$ and $y = b$. Then the area is $\int_a^b (f(y) - g(y))dy$ (see the remark in 1).

Both approaches can be combined with each other and with formulas from elementary geometry (when some of the regions are triangles or rectangulars).

Important Remark: Draw a picture to visualize the region! The condition $f(x) \geqslant g(x)$ in the integral formulas is important: otherwise some parts of the region will contribute to the area with the minus sign. In fact, the formula is $\int_a^b |f(x) - g(x)|\,dx$ (see remark in 3).

5.2. Arc length. The length of a curve is given by $L = \int ds$, where the *arc length element* ds is given by $ds = \sqrt{dx^2 + dy^2}$ (infinitesimal 'Pythagorean theorem'). Here are special cases:

The graph $y = f(x)$, $a \leqslant x \leqslant b$: $$L = \int_a^b \sqrt{1 + (f')^2}dx.$$

The graph $x = g(y)$, $a \leqslant y \leqslant b$: $$L = \int_a^b \sqrt{1 + (g')^2}dy.$$

Parametric representation $x = x(t)$, $y = y(t)$, $\alpha \leqslant t \leqslant \beta$: $$L = \int_\alpha^\beta \sqrt{(x')^2 + (y')^2}dt.$$

5.3. Volume of a solid of revolution. The volume of the solid generated by revolving the region bounded by the graph of a function $f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by:

$$V = \int_a^b \pi f^2(x)\,dx \quad (\textit{disk method}; \text{ rotation about the } x\text{-axis}), \text{ or}$$

$$V = \int_a^b 2\pi x f(x)\,dx \quad (\textit{shell method}; \text{ rotation about the } y\text{-axis; must have } f(x) \geqslant 0).$$

In the former case (disk method), the region is sliced into **vertical** segments (**perpendicular** to the axis of revolution); the method applies to revolution about any **horizontal** axes. The region must be made of vertical segments **with one end on the axis of revolution**, and one has

$$V = \int_a^b \pi R^2 dx,$$

where $R = R(x)$ is the length of the vertical segment through x.

In the latter case (shell method), the region is also sliced into **vertical** segments (which are now **parallel** to the axis of revolution); the method applies to revolution about any **vertical** axes. The formula takes the form

$$V = \int_a^b 2\pi d(x) l(x)\,dx,$$

where $l(x)$ is the length of the vertical segment through x and $d(x)$ is its distance from the axis of revolution. Make sure that $l(x)$ and $d(x)$ are **positive**.

Important Remark: Each method has its own advantages and disadvantages. In the **disk method**, the sign of $R(x)$ is not important, but the region should be 'adjacent' to the axis of revolution. If there are 'holes', the volume is found as difference/sum of volumes of simpler solids. In the **shell method**, the region is arbitrary, but one should make sure that the expressions for $l(x)$ and $d(x)$ are positive (see remark in 3).

Important Remark: In both cases, **avoid overlaps**, which may result from a region 'intersecting' the axis of revolution.

A more complicated region should be subdivided into simple ones as when calculating areas. Of course, the disk method can be applied to a vertical axis and the shell method can be applied to a horizontal axis; in this case, one slices the region into **horizontal** segments and integrates with respect to y.

5.4. Area of a surface of revolution. The area of the surface generated by revolving a curve $y = f(x)$, $a \leqslant x \leqslant b$, is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f')^2}dx \quad (\text{rotation about the } x\text{-axis}), \text{ or}$$

$$S = \int_a^b 2\pi x\sqrt{1 + (f')^2}dx \quad (\text{rotation about the } y\text{-axis}).$$

The general formula (for arbitrary curve) is

$$S = \int_a^b r\,ds,$$

where r is the distance from a point of the curve to the axis of revolution and ds is the arc length element at this point (see arc length).

5.5. General volume formula. The volume of a solid is given by $V = \int_a^b S(x)\,dx$, where $S(x)$ is the area of the cross-section through x perpendicular to the x-axis and a and b are the x-coordinates of, respectively, the leftmost and the rightmost points of the solid.

6. Applications to physics [advanced]

6.1. Work. The work of a force $F(x)$ directed along the x-axis on the segment $a \leqslant x \leqslant b$ is $W = \int_a^b F(x)\,dx$.

6.2. Fluid force. The total force of a fluid of density w against one side of a submerged vertical plate running from depth $y = a$ to depth $y = b$ is $F = \int_a^b wyL(y)\,dy$, where $L(y)$ is the horizontal extent of the plate (measured along the plate) at depth y. (In this formula the y-axis is assumed to point downwards.)

6.3. Moments, mass, center of gravity. The coordinates (x_c, y_c) of the center of gravity of a system are given by $x_c = M_x/M$, $y_c = M_y/M$, where M is the mass of the system and M_x, M_y are its moments about the y-axis and x-axis, respectively. Here are some special cases:

Thin rod along the x-axis of linear density $\delta(x)$.
$M = \int_a^b \delta(x)\,dx, \quad M_x = \int_a^b x\delta(x)\,dx, \quad M_y = 0.$

Thin wire along the graph $y = f(x)$, $a \leqslant x \leqslant b$, of linear density $\delta(x)$.
$M = \int_a^b \delta(x)\sqrt{1+(f')^2}dx, \quad M_x = \int_a^b x\delta(x)\sqrt{1+(f')^2}dx, \quad M_y = \int_a^b f(x)\delta(x)\sqrt{1+(f')^2}dx.$
More generally, the formulas are
$M = \int_a^b \delta\,ds, \quad M_x = \int_a^b x\delta\,ds, \quad M_y = \int_a^b y\delta\,ds$, see *arc length*.

Thin flat plate bounded by the graphs of $f(x) \geqslant g(x)$ and the vertical lines $x = a$, $x = b$, of surface density $\delta(x, y)$. [1]
$M = \int_a^b \big(\int_{g(x)}^{f(x)} \delta(x,y)\,dy\big)dx, \quad M_x = \int_a^b \big(\int_{g(x)}^{f(x)} x\delta(x,y)\,dy\big)dx, \quad M_y = \int_a^b \big(\int_{g(x)}^{f(x)} y\delta(x,y)\,dy\big)dx.$

Thin flat plate bounded by the curves $x = f(y)$, $x = g(y)$, $f(y) \geqslant g(y)$, and the horizontal lines $y = a$, $y = b$, of surface density $\delta(x, y)$. [1]
$M = \int_a^b \big(\int_{g(y)}^{f(y)} \delta(x,y)\,dx\big)dy, \quad M_x = \int_a^b \big(\int_{g(y)}^{f(y)} x\delta(x,y)\,dx\big)dy, \quad M_y = \int_a^b \big(\int_{g(y)}^{f(y)} y\delta(x,y)\,dx\big)dy.$

In the last two cases be **very** careful about the variable in respect to which you integrate! (See remark in 1.) Say, in the last case, after the inner integral (with respect to x, with y treated as a constant) is evaluated, the expression must no longer contain x, and the outer integration is in respect to y.

More complicated regions should be subdivided into simple ones of one of the two above forms. Mass and moments are additive. Keep in mind that mass must always be positive, while moments can take negative values as well.

Important Remark: Use symmetry whenever possible! If both the plate and the density are symmetric with respect to an axis, the corresponding moment is 0.

7. Techniques of integration

See `integration.pdf`.

8. Improper integrals [advanced]

A definite integral is called *improper* if either it has infinite limits or the integrand is discontinuous (or both). To evaluate an improper integral, split the interval of integration into subintervals so that each subinterval has at most one singularity (i.e., an infinite endpoint or a point of discontinuity of the integrand, which must coincide with one of the endpoints of the subinterval).

8.1. Example. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x}$ should be split into $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x} + \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} + \int_{1}^{\infty} \frac{dx}{x}$. The subdivision points $-1 \in (-\infty, 0)$ and $1 \in (0, \infty)$ are chosen arbitrarily.

[1] In fact, these are so called *double integrals*; they are considered in Math 102

Now an integral with one singularity is defined as follows:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \text{(infinite upper limit)},$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \qquad \text{(infinite lower limit)},$$
$$\int_a^b f(x)\,dx = \lim_{b'\to b^-}\int_a^{b'} f(x)\,dx \qquad \text{($f(x)$ is discontinuous at b)},$$
$$\int_a^b f(x)\,dx = \lim_{a'\to a^+}\int_{a'}^b f(x)\,dx \qquad \text{($f(x)$ is discontinuous at a)}.$$

If the limit exists, the integral is said to *converge*; otherwise it is said to *diverge*. An integral with several singularities converges if and only if so does **each** of the integrals over the subintervals.

8.2. Example. $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\Big(1-\frac{1}{b}\Big) = 1$ converges; $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0^+} -\frac{1}{x}\Big|_a^1 = \infty$ diverges.

Important Remark: Do **not** try to apply the Newton-Leibniz formula $\int_a^b f(x)\,dx = F(x)\big|_a^b$ to discontinuous functions! First, it doesn't make sense, second, it may give a wrong result. For example, $\int_{-1}^1 dx/x = \int_{-1}^0 dx/x + \int_0^1 dx/x$ diverges (as $\lim_{x\to 0^+}\ln x = -\infty$), while the Newton-Leibniz formula would give 0.

In many cases it is only important to know whether the integral converges or not. The following remarks/tests may help to decide this. (For simplicity we consider singularity at infinity; the limit ∞ below can be replaced with a point of discontinuity of f.)

8.3. Independence of the limits. The integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ converge or diverge simultaneously provided that f is continuous on $[a,b]$. (Of course, the **value** of the integral **does** depend on the limits.)

8.4. The domination test. Assume that $0 \leqslant f(x) \leqslant g(x)$ for all sufficiently large x. Then

(1) if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$, and
(2) if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.

8.5. The limit comparison test. Assume that $f(x)$ and $g(x)$ are positive functions and

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = L, \quad 0 < L < \infty.$$

Then $\int_a^\infty g(x)\,dx$ and $\int_a^\infty f(x)\,dx$ converge or diverge simultaneously.

The following integrals are useful for comparison:

$$\int_1^\infty \frac{dx}{x^p} \qquad \text{converges for } p>1 \text{ and diverges for } p\leqslant 1,$$
$$\int_0^1 \frac{dx}{x^p} \qquad \text{converges for } p<1 \text{ and diverges for } p\geqslant 1,$$
$$\int_0^\infty e^{ax}\,dx \qquad \text{converges for } a<0 \text{ and diverges for } a\geqslant 0.$$

[advanced] This topic has been omitted or moved to Math 102