Solutions to Midterm II

Problem 1. Find the limits:

(a) $\lim_{x\to 0^+} x^x$

(b) $\lim_{x\to 0}(e^x+x)^{1/x}$

(c) $\lim_{x\to 0}\dfrac{x(1-\cos x)}{x-\sin x}$.

SOLUTION: (mainly using l'Hôpital's rule)

(a) $\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x} = e^0 = \boxed{1}$ since $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\dfrac{\ln x}{1/x} = \left[\dfrac{0}{0}\right] = \lim_{x\to 0^+}\dfrac{1/x}{-1/x^2} = \lim_{x\to 0^+} x = 0.$

(b) $\lim_{x\to 0}(e^x+x)^{1/x} = e^{\lim_{x\to 0}\ln(e^x+x)/x} = \boxed{e^2}$, since $\lim_{x\to 0}\dfrac{\ln(e^x+x)}{x} = \left[\dfrac{0}{0}\right] = \lim_{x\to 0}\dfrac{e^x+1}{e^x+x} = 2.$

(c) $\lim_{x\to 0}\dfrac{x(1-\cos x)}{x-\sin x} = \left[\dfrac{0}{0}\right] = \lim_{x\to 0}\dfrac{(1-\cos x)+x\sin x}{1-\cos x} = \left[\dfrac{0}{0}\right] = \lim_{x\to 0}\dfrac{\sin x+\sin x+x\cos x}{\sin x} = \boxed{3}.$

Remark. A common mistake was to misuse l'Hôpital's rule in the form $\lim f(x) = \lim f'(x)$. **This is wrong!** Another mistake is to use the rule in the (correct) form $\lim f(x)/g(x) = \lim f'(x)/g'(x)$ when the condition $\lim f(x) = \lim g(x) = 0$ or ∞ fails.

Problem 2. Find the center of mass of a thin plate of constant density δ covering the region bounded by the parabola $x = y^2 - y$ and the line $y = x$.

SOLUTION: The points of intersection of the two curves, found from the equation $y^2 - y = y$, are $y = 0$ and $y = 2$. Note also that on the interval $[0, 2]$ one has $y^2 - y \le y$. (Just draw a picture.) Thus, the mass and the moments of the plate are

$$M = \int_0^2\left(\int_{y^2-y}^{y}\delta\,dx\right)dy = \int_0^2 \delta(2y-y^2)\,dy = \delta\left(y^2-\frac{y^3}{3}\right)\Big|_0^2 = \frac{4\delta}{3},$$

$$M_x = \int_0^2\left(\int_{y^2-y}^{y}\delta x\,dx\right)dy = \int_0^2 \frac{\delta}{2}[y^2-(y^2-y)^2]\,dy = \delta\left(\frac{1}{4}y^4-\frac{y^5}{10}\right)\Big|_0^2 = \frac{4\delta}{5},$$

$$M_y = \int_0^2\left(\int_{y^2-y}^{y}\delta y\,dx\right)dy = \int_0^2 \delta(2y-y^2)y\,dy = \delta\left(\frac{2}{3}y^3-\frac{y^4}{4}\right)\Big|_0^2 = \frac{4\delta}{3}.$$

Thus, the coordinates $(\bar{x},\bar{y})$ of the center of mass are $\bar{x} = M_x/M = \boxed{3/5}$ and $\bar{y} = M_y/M = \boxed{1}$.

Remark. Note that, unlike most other problems, here we treat x and y as **independent** variables, i.e., when integrating with respect to x, we assume that $y = \text{const}$.

Problem 3. (a) Find $\dfrac{d}{dx}(\sin x)^{\tan x}$.

(b) Find $\lim_{x\to\infty}\dfrac{1}{x}\displaystyle\int_0^x \frac{t^4dt}{(1+t^2)^2}$.

SOLUTION: (a) Let $y = (\sin x)^{\tan x}$. Then $\ln y = \tan x \ln\sin x$. Hence,

$$\frac{y'}{y}(\ln y)' = \frac{1}{\cos^2 x}\ln\sin x + \tan x\frac{\cos x}{\sin x} = \frac{\ln\sin x}{\cos^2 x}+1 \qquad\text{and}\qquad \boxed{y' = (\sin x)^{\tan x}\left(\frac{\ln\sin x}{\cos^2 x}+1\right)}.$$

Remark. A common mistake was to use the formula $(x^n)' = nx^{n-1}$ or $(a^x)' = a^x\ln a$. **Neither** of them applies directly!

(b) First, notice that the integral $\int_0^\infty t^4dt/(1+t^2)^2$ diverges (limit comparison test with $\int_0^\infty dt = \infty$). Hence, l'Hôpipal's rule applies to yield $\lim_{x\to\infty} x^4/(1+x^2)^2 = \boxed{1}$. Alternatively, one can notice that $\int_0^x t^4dt/(1+t^2)^2 = x - \int_0^x(2t^2+1)dt/(1+t^2)^2$ and that the integral $\int_0^\infty(2t^2+1)dt/(1+t^2)^2$ converges (compare with $\int_1^\infty dt/t^2$). Thus, the limit is $1 - \lim_{x\to\infty}\{\text{bounded function}\}/x = 1 - 0 = \boxed{1}$.

Problem 4. The region between the curve $y = x\ln x$ and the x-axis from $x = 0$ to $x = 1$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

SOLUTION: Just use the formula:

$$V = \int_0^1 \pi y^2\, dx = \pi \int_0^1 x^2 (\ln x)^2 dx = \begin{bmatrix} u = (\ln x)^2 & v = x^3/3 \\ du = 2\ln x/x & dv = x^2 \end{bmatrix} = \frac{\pi}{3} x^3 (\ln x)^2 \Big|_0^1 - \frac{2\pi}{3}\int_0^1 x^2 \ln x\, dx$$

$$= \begin{bmatrix} u = \ln x & v = x^3/3 \\ du = 1/x & dv = x^2 \end{bmatrix} = 0 - \frac{2\pi}{3}\cdot\frac{1}{3} x^3 \ln x \Big|_0^1 + \frac{2\pi}{3}\cdot\frac{1}{3}\int_0^1 x^2\, dx = \boxed{\frac{2\pi}{27}}.$$

Remark. Note that the integral is, in fact, proper as $\lim_{x\to 0} x \ln x = 0$, cf. **1**(a). In particular, both the exintegral terms are 0. However, this needs proof, as you get $[0 \cdot \infty]$.

Problem 5. Evaluate:

(a) $\displaystyle\int \frac{x^3 - x}{(x^2+1)(x-1)^2}\, dx$

(b) $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

SOLUTION: (a) $\dfrac{x^3 - x}{(x^2+1)(x-1)^2} = \dfrac{x^2 + x}{(x^2+1)(x-1)} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2+1}$. Multiplying by $(x-1)(x^2+1)$, we get the identity $x^2 + x = (A+B)x^2 + (-B+C)x + (A-C)$. Hence, $A = C = 1$ and $B = 0$ and the integral is $\displaystyle\int \frac{dx}{x-1} + \int \frac{dx}{x^2+1} = \boxed{\ln|x-1| + \tan^{-1} x + C}$.

(b) Let's use the universal trigonometric substitution $t = \tan(x/2)$. Then the integral is

$$\int \frac{2dt/(1+t^2)}{1 + (2t + 1 - t^2)/(1+t^2)} = \int \frac{dt}{t+1} = \ln|t+1| = \boxed{\ln\left|\tan\frac{x}{2} + 1\right| + C}.$$