Summary: Deformation classes

In this section we state our principal results concerning real Enriques surfaces and explain how the proofs are distributed among the chapters.

As is known, the topology of a real structure and, more generally, of a Klein action is invariant under deformation (see 7.5.1). It turns out that for real Enriques surfaces the converse also holds: *the deformation type of a real Enriques surface E is determined by the topology of its real structure.* Moreover, the latter is determined by the induced $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action in the homology $H_2(X;\mathbb{Z})$ of the covering $K3$-surface, see 14.2.3. The actions and, thus, the deformation classes, are enumerated in Appendix C.

Below we reinterpret the above statement in more visual terms: the real part $E_{\mathbb{R}}$, the half decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$ (see 9.2), types of $E_{\mathbb{R}}$ and $E_{\mathbb{R}}^{(i)}$ in E and $X/t^{(i)}$ (see the beginning of Chapter IV and 9.2), and the Pontrjagin-Viro form and its invariants, such as various versions of complex separation (see 2.5 and 9.8.2). The relation between these invariants and the induced homology action is explained in corresponding proofs; a common table is found in Appendix C.

To enumerate the topological types of the real part we use the notion of *topological Morse simplification*, *i.e.*, Morse transformation of the topological type which decreases the total Betti number. Thus, a topological Morse simplification is either removing a spherical component ($S \to \varnothing$) or contracting a handle ($S_{g+1} \to S_g$ or $V_{p+2} \to V_p$).

The complex deformation type of surfaces being fixed (*e.g.*, $K3$ or Enriques), a topological type (*i.e.*, a class of surfaces with homeomorphic real parts) is called *extremal* if it cannot be obtained from another one (in the same complex deformation type) by a topological Morse simplification. Note that a topological Morse simplification may not correspond to a Morse simplification in a continuous family of surfaces. As a result, the notions of extremal topological type and extremal (in the obvious sense) surface may be different. *E.g.*, as it was observed by Viro and Kharlamov [Vir2], any surface whose real part is mod 2 homologous to zero in its complexification (*i.e.*, type I_0) is extremal, even though it may have nonextremal topological type.

We believe that in the case of real Enriques surfaces any topological Morse simplification can be realized in a family of algebraic surfaces. However, we did not check this assertion thoroughly. A partial statement is found in 16.3.

topology **S1. Topology of the real part.** *There are* 87 *topological types of the real parts of real Enriques surfaces. Each of them can be obtained by a sequence of topological Morse simplifications from one of the* 22 *extremal types listed below. Conversely, with the exception of the two types* $6S$ *and* $S_1 \sqcup 5S$, *any topological type obtained in this way is realized by a real Enriques surface.*

The 22 *extremal types are*:

(1) M*-surfaces*:

(a) $\chi(E_{\mathbb{R}}) = 8$:

$4V_1 \sqcup 2S$, $V_2 \sqcup 2V_1 \sqcup 3S$,
$V_3 \sqcup V_1 \sqcup 4S$, $2V_2 \sqcup 4S$,
$V_4 \sqcup 5S$, $V_2 \sqcup S_1 \sqcup 4S$,

(b) $\chi(E_{\mathbb{R}}) = -8$:

$V_{11} \sqcup V_1$, $V_{10} \sqcup V_2$,
$V_9 \sqcup V_3$, $V_8 \sqcup V_4$,
$V_7 \sqcup V_5$, $2V_6$,
$V_{10} \sqcup S_1$;

(2) $(M-2)$*-surfaces with* $\chi(E_{\mathbb{R}}) = 0$:

$V_4 \sqcup 2V_1$, $V_3 \sqcup V_2 \sqcup V_1$, $V_5 \sqcup V_1 \sqcup S$, $V_4 \sqcup V_2 \sqcup S$,
$V_6 \sqcup 2S$, $V_4 \sqcup S_1 \sqcup S$, $2V_3 \sqcup S$, $2V_2 \sqcup S_1$;

(3) *Pair of tori*: $2S_1$.

halves **S2. Half decomposition.** *Each half of a real Enriques surface is either*

halves-i1 (1) $\alpha V_g \sqcup a V_1 \sqcup bS$ *with* $g > 1$, $a \geqslant 0$, $b \geqslant 0$, $\alpha = 0, 1$, *or*

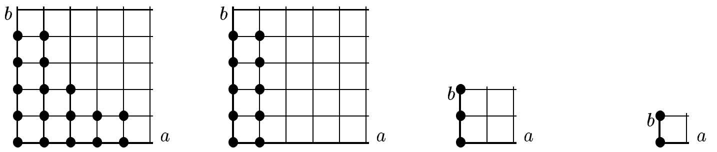
halves-i2 (2) $2V_2$, *or*

halves-i3 (3) S_1.

With the exception of the cases $E_{\mathbb{R}} = kS$ *and* $E_{\mathbb{R}} = V_{2r} \sqcup kS$ *a half decomposition of a real part* $E_{\mathbb{R}}$ *as in* S1 *is realizable if and only if it satisfies* (1)–(3) *above. The realizable half decompositions of the exceptional real parts are listed in Figure* 4.

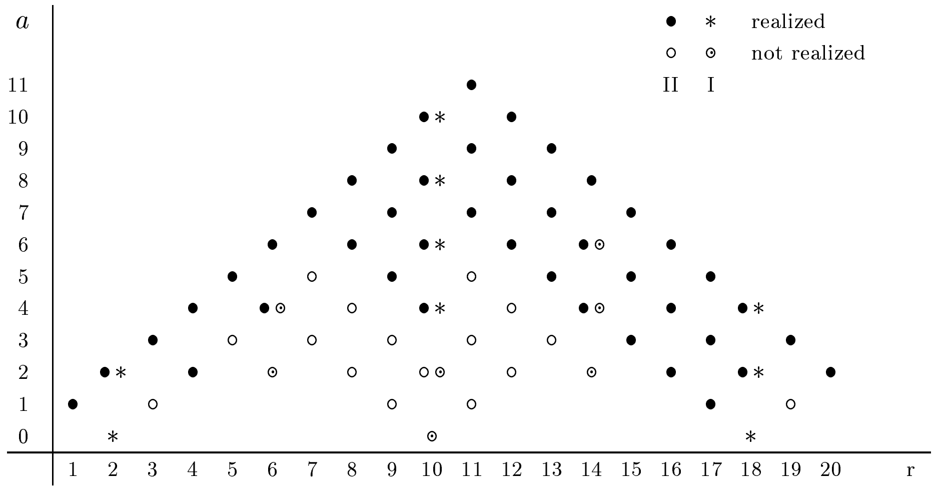
K3.halves **S3. $K3$-surfaces covering real Enriques surfaces.** *The real* $K3$*-surfaces appearing as double coverings of real Enriques surfaces are those listed in Figure* 5.

For a technical reason we divide (flexible) real Enriques surfaces into three classes. A real Enriques surface E is said to be of *hyperbolic*, *parabolic*, or *elliptic* type if the minimal Euler characteristic of the components of the real part $E_{\mathbb{R}}$ is negative, zero, or positive, respectively.



$\{aS\}\sqcup\{bS\}$, $\{V_2\sqcup aS\}\sqcup\{bS\}$ — $\{V_4\sqcup aS\}\sqcup\{bS\}$ — $\{V_6\sqcup aS\}\sqcup\{bS\}$ — $\{V_8\sqcup aS\}\sqcup\{bS\}$, $\{V_{10}\sqcup aS\}\sqcup\{bS\}$

halves.table FIGURE 4. Half decompositions of $E_{\mathbb{R}} = kS$ and $E_{\mathbb{R}} = V_{2r} \sqcup kS$ (the lower left corner of each diagram is the origin $a = b = 0$)



$*$ and $\bullet$ denote the $K3$-surfaces of type I and II, respectively, realized as $X_{\mathbb{R}}^{(i)}$; $\odot$ and $\circ$ denote the surfaces of type I and II, respectively, not realized as $X_{\mathbb{R}}^{(i)}$. The parameters are $r = 10 + \frac{1}{2}\chi(X_{\mathbb{R}})$ and $a = 12 - \frac{1}{2}\beta_*(X_{\mathbb{R}})$ for $X_{\mathbb{R}} \neq \varnothing$ or $a = 10$ for $X_{\mathbb{R}} = \varnothing$. (Recall that a $K3$-surface X is determined up to deformation by its type, a, and r, see 8.4.2.)

K3.E FIGURE 5. Real $K3$-surfaces appearing as $X_{\mathbb{R}}^{(i)}$

S.def-cl **S4. Deformation classes.** *With few exceptions listed below the deformation class of a real Enriques surface E with a distinguished half $E_{\mathbb{R}}^{(1)}$ is determined by the topology of its half decomposition. The exceptions are*:

S.def-cl-i1 (1) *M-surfaces of parabolic and elliptic type*, i.e., *those with $E_{\mathbb{R}} = 2V_2 \sqcup 4S$, $V_2 \sqcup 2V_1 \sqcup 3S$, or $4V_1 \sqcup 2S$; the additional invariant is the Pontrjagin-Viro form, see* S5 *below*;

S.def-cl-i2 (2) *surfaces with $E_{\mathbb{R}} = 2V_1 \sqcup 4S$; the additional invariant is the integral complex separation (see* 22.4.1*)*;

(3) *surfaces with a half $E_{\mathbb{R}}^{(1)} = 4S$ other than those mentioned in* (1), (2); *the additional invariants are the types,* I_u, I_0, *or* II, *of $E_{\mathbb{R}}^{(1)}$ in E and $X/t^{(2)}$*;

(4) *surfaces with $E_{\mathbb{R}} = \{V_{10}\} \sqcup \{\varnothing\}$, $\{V_4 \sqcup S\} \sqcup \{\varnothing\}$, $\{V_2 \sqcup 4S\} \sqcup \{\varnothing\}$, and $\{2S\} \sqcup \{2S\}$; the additional invariant is the type,* I_u *or* I_0, *of $E_{\mathbb{R}}$ in E*;

(5) *surfaces with* $E_{\mathbb{R}} = 2V_1 \sqcup 3S$; *the additional invariant is the type,* I_u *or* II, *of* $E_{\mathbb{R}}$ *in* E;
(6) *surfaces with* $E_{\mathbb{R}} = \{S_1\} \sqcup \{S_1\}$; *the additional invariant is the linking coefficient form of* $E_{\mathbb{R}}^{(1)}$ (*see* 20.1).

A complete list of deformation classes, as well as the invariants necessary to distinguish them, are found in Appendix C.

S.pvf **S5. The Pontrjagin-Viro form.** *With the exceptions listed below the Pontrjagin-Viro form on a real Enriques surface* E *is determined up to isomorphism by the decomposition* $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$. *The exceptions are*:

(1) *surfaces with* $E_{\mathbb{R}} = 4V_1 \sqcup 2S$: *the Pontrjagin-Viro form is determined by the complex separation* (*see* 9.8.2);
(2) *surfaces with* $E_{\mathbb{R}} = 2V_2 \sqcup 4S$ *and* $E_{\mathbb{R}} = V_2 \sqcup 2V_1 \sqcup 3S$: *the Pontrjagin-Viro form is determined by the complex separation and the value of the form on the characteristic class of a component* V_2.

Moreover, given a decomposition $E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ *as in* S2, *any quadratic form* $\mathcal{P}$ *on* $H_*(E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)})$ *satisfying* 9.8.5 *can be realized as the Pontrjagin-Viro form of a real Enriques surface. The deformation types* (*up to permutation of the halves and quoters*) *of surfaces admitting Pontrjagin-Viro form are listed in Table* 5.

The topological part of the statements above, namely, the restrictions to the topology of the real part $E_{\mathbb{R}}$ and its halves $E_{\mathbb{R}}^{(i)}$, the topological (and, hence, deformation) type of the covering $K3$-surface, and the values of the Pontrjagin-Viro form, are proved in 11.3 in Chapter V. The methods are purely topological and apply to flexible and generalized real Enriques surfaces as well; the primary tools are those developed in Chapter I.

The deformation classification (proof of Theorem S4) is dealt with in Chapters VII and VIII. More precisely, in Chapter VII we treat all surfaces of hyperbolic type and most surfaces of parabolic type (except those with $E_{\mathbb{R}}^{(1)} = V_2$). The methods are geometric: we use Donaldson's trick (Chapter VI), reduce the problem to a question on nonsingular anti-bicanonical curves on rational surfaces, and construct convenient models of the resulting rational surfaces. This leads to a number of various classification problems related to real rational surfaces and curves on them; they are discussed in Appendix A. In Chapter VIII we consider the remaining cases, *i.e.*, surfaces of elliptic type and those with $E_{\mathbb{R}}^{(1)} = V_2$. (Some other surfaces are also re-considered there in order to express their topological invariants used in the classification in terms of the homological type of the action.) The methods are arithmetical and rely upon the tools developed in Chapter II.

ts.pvf

TABLE 5. The Pontrjagin-Viro form

$E_{\mathbb{R}}$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$
$V_4 \sqcup 5S$	$(V_4^0 \sqcup 3S) \sqcup (2S)$	$\varnothing$
	$(V_4^0 \sqcup 2S) \sqcup (S)$	$(S) \sqcup (S)$
	$(V_4^0 \sqcup S) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
$V_3 \sqcup V_1 \sqcup 4S$	$(V_3^1 \sqcup V_1^{-1} \sqcup 2S) \sqcup (2S)$	$\varnothing$
	$(V_3^1 \sqcup 2S) \sqcup (2S)$	$(V_1^{-1}) \sqcup (\varnothing)$
	$(V_3^1 \sqcup 2S) \sqcup (V_1^1 \sqcup S)$	$(S) \sqcup (\varnothing)$
	$(V_3^{-1} \sqcup 2S) \sqcup (S)$	$(V_1^1) \sqcup (S)$
	$(V_3^1 \sqcup V_1^{-1} \sqcup S) \sqcup (S)$	$(S) \sqcup (S)$
	$(V_3^1 \sqcup S) \sqcup (S)$	$(V_1^{-1} \sqcup S) \sqcup (S)$
	$(V_3^1 \sqcup S) \sqcup (V_1^1)$	$(2S) \sqcup (S)$
	$(V_3^{-1} \sqcup S) \sqcup (\varnothing)$	$(V_1^1 \sqcup S) \sqcup (2S)$
	$(V_3^1 \sqcup V_1^{-1}) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
	$(V_3^1) \sqcup (\varnothing)$	$(V_1^{-1} \sqcup 2S) \sqcup (2S)$
$2V_2 \sqcup 4S$	$(V_2^0 \sqcup 2S) \sqcup (2S)$	$(V_2^0) \sqcup (\varnothing)$
	$(V_2^{-2} \sqcup 2S) \sqcup (S)$	$(V_2^2) \sqcup (S)$
	$(V_2^2 \sqcup S) \sqcup (2S)$	$(V_2^{-2} \sqcup S) \sqcup (\varnothing)$
	$(V_2^0 \sqcup S) \sqcup (S)$	$(V_2^0 \sqcup S) \sqcup (S)$
	$(2V_2^0) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
	$(V_2^2 \sqcup V_2^{-2}) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
	$(V_2^2) \sqcup (V_2^2)$	$(3S) \sqcup (S)$
	$(V_2^0) \sqcup (V_2^0)$	$(2S) \sqcup (2S)$
$V_2 \sqcup 2V_1 \sqcup 3S$	$(V_2^0 \sqcup V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (2S)$	$\varnothing$
	$(V_2^2 \sqcup 2V_1^{-1} \sqcup S) \sqcup (2S)$	$\varnothing$
	$(V_2^0 \sqcup 2S) \sqcup (V_1^1 \sqcup V_1^{-1} \sqcup S)$	$\varnothing$
	$(V_2^0 \sqcup V_1^1 \sqcup S) \sqcup (2S)$	$(V_1^{-1}) \sqcup (\varnothing)$
	$(V_2^2 \sqcup V_1^{-1} \sqcup S) \sqcup (2S)$	$(V_1^{-1}) \sqcup (\varnothing)$
	$(V_2^0 \sqcup 2S) \sqcup (V_1^1 \sqcup S)$	$(V_1^1) \sqcup (\varnothing)$
	$(V_2^0 \sqcup V_1^1 \sqcup S) \sqcup (V_1^1 \sqcup S)$	$(S) \sqcup (\varnothing)$
	$(V_2^2 \sqcup V_1^{-1} \sqcup S) \sqcup (V_1^1 \sqcup S)$	$(S) \sqcup (\varnothing)$
	$(V_2^0 \sqcup 2S) \sqcup (S)$	$(V_1^1) \sqcup (V_1^1)$
	$(V_2^2 \sqcup S) \sqcup (2S)$	$(2V_1^{-1}) \sqcup (\varnothing)$
	$(V_2^0 \sqcup S) \sqcup (2S)$	$(V_1^{-1}) \sqcup (V_1^{-1})$
	$(V_2^0 \sqcup V_1^{-1} \sqcup S) \sqcup (S)$	$(V_1^1) \sqcup (S)$
	$(V_2^{-2} \sqcup V_1^1 \sqcup S) \sqcup (S)$	$(V_1^1) \sqcup (S)$
	$(V_2^2 \sqcup S) \sqcup (V_1^1 \sqcup S)$	$(V_1^{-1} \sqcup S) \sqcup (\varnothing)$
	$(V_2^0 \sqcup S) \sqcup (V_1^{-1} \sqcup S)$	$(V_1^{-1}) \sqcup (S)$

Comments to the table. For each topological type of $E_{\mathbb{R}}$ the table lists the decompositions (up to permutation) $E_{\mathbb{R}}^{(1)} = (Q_1^{(1)}) \sqcup (Q_2^{(1)})$ and $E_{\mathbb{R}}^{(2)} = (Q_1^{(2)}) \sqcup (Q_2^{(2)})$ of the halves into quarters, see 9.8.2. The upper index b of a nonspherical component $C \subset Q_j^{(i)}$, $i, j = 1, 2$, is the Brown invariant of the restriction to $H_1(C)$ of $\mathfrak{q}_{j1}^{(i)}$, see 9.8.4. If $E_{\mathbb{R}}^{(2)} = \varnothing$ (and, thus, $\mathfrak{q}_{jk}^{(i)}$ are not defined), b is the Brown invariant of the restriction to $H_1(C)$ of (any) extension of $\mathcal{P}$ to $H_1(E_{\mathbb{R}}^{(1)})$.

TABLE 5 (continued)

$E_{\mathbb{R}}$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$
(continued)	$(V_2^0 \sqcup V_1^1 \sqcup V_1^{-1}) \sqcup (S)$	$(S) \sqcup (S)$
	$(V_2^2 \sqcup 2V_1^{-1}) \sqcup (S)$	$(S) \sqcup (S)$
	$(V_2^2 \sqcup S) \sqcup (2V_1^1)$	$(2S) \sqcup (\varnothing)$
	$(V_2^0 \sqcup S) \sqcup (V_1^1 \sqcup V_1^{-1})$	$(S) \sqcup (S)$
	$(V_2^0 \sqcup S) \sqcup (S)$	$(V_1^1 \sqcup V_1^{-1}) \sqcup (S)$
	$(V_2^2 \sqcup S) \sqcup (S)$	$(V_1^{-1} \sqcup S) \sqcup (V_1^1)$
	$(V_2^0 \sqcup V_1^1) \sqcup (S)$	$(V_1^{-1} \sqcup S) \sqcup (S)$
	$(V_2^2 \sqcup V_1^{-1}) \sqcup (S)$	$(V_1^{-1} \sqcup S) \sqcup (S)$
	$(V_2^0 \sqcup S) \sqcup (V_1^1)$	$(V_1^1 \sqcup S) \sqcup (S)$
	$(V_2^{-2} \sqcup S) \sqcup (V_1^{-1})$	$(V_1^1) \sqcup (2S)$
	$(V_2^0 \sqcup V_1^1) \sqcup (V_1^1)$	$(2S) \sqcup (S)$
	$(V_2^2 \sqcup V_1^{-1}) \sqcup (V_1^1)$	$(2S) \sqcup (S)$
	$(V_2^{-2} \sqcup S) \sqcup (\varnothing)$	$(2V_1^1) \sqcup (2S)$
	$(V_2^0 \sqcup S) \sqcup (\varnothing)$	$(V_1^1 \sqcup S) \sqcup (V_1^1 \sqcup S)$
	$(V_2^2) \sqcup (S)$	$(2V_1^{-1} \sqcup S) \sqcup (S)$
	$(V_2^0) \sqcup (S)$	$(V_1^{-1} \sqcup S) \sqcup (V_1^{-1} \sqcup S)$
	$(V_2^0 \sqcup V_1^{-1}) \sqcup (\varnothing)$	$(V_1^1 \sqcup S) \sqcup (2S)$
	$(V_2^{-2} \sqcup V_1^1) \sqcup (\varnothing)$	$(V_1^1 \sqcup S) \sqcup (2S)$
	$(V_2^2) \sqcup (V_1^1)$	$(V_1^{-1} \sqcup 2S) \sqcup (S)$
	$(V_2^0) \sqcup (V_1^{-1})$	$(V_1^{-1} \sqcup S) \sqcup (2S)$
	$(V_2^0) \sqcup (\varnothing)$	$(V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (2S)$
	$(V_2^2) \sqcup (\varnothing)$	$(V_1^{-1} \sqcup 2S) \sqcup (V_1^1 \sqcup S)$
$V_2 \sqcup S_1 \sqcup 4S$	$(V_2^0 \sqcup 2S) \sqcup (2S)$	$(S_1^0) \sqcup (\varnothing)$
$V_2 \sqcup 4S$	$(V_2^0 \sqcup 2S) \sqcup (2S)$	$\varnothing$
	$(V_2^0 \sqcup S) \sqcup (S)$	$(S) \sqcup (S)$
	$(V_2^0) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
$4V_1 \sqcup 2S$	$(2V_1^1 \sqcup 2V_1^{-1}) \sqcup (2S)$	$\varnothing$
	$(V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (V_1^1 \sqcup V_1^{-1} \sqcup S)$	$\varnothing$
	$(2V_1^1 \sqcup V_1^{-1}) \sqcup (2S)$	$(V_1^{-1}) \sqcup (\varnothing)$
	$(V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (V_1^1 \sqcup S)$	$(V_1^1) \sqcup (\varnothing)$
	$(2V_1^1 \sqcup V_1^{-1}) \sqcup (V_1^1 \sqcup S)$	$(S) \sqcup (\varnothing)$
	$(V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (S)$	$(V_1^1) \sqcup (V_1^1)$
	$(2V_1^1) \sqcup (2S)$	$(2V_1^{-1}) \sqcup (\varnothing)$
	$(V_1^1 \sqcup V_1^{-1}) \sqcup (2S)$	$(V_1^{-1}) \sqcup (V_1^{-1})$
	$(V_1^1 \sqcup S) \sqcup (V_1^1 \sqcup S)$	$(V_1^1 \sqcup V_1^{-1}) \sqcup (\varnothing)$
	$(V_1^1 \sqcup S) \sqcup (V_1^{-1} \sqcup S)$	$(V_1^{-1}) \sqcup (V_1^1)$
	$(V_1^1 \sqcup 2V_1^{-1}) \sqcup (S)$	$(V_1^1) \sqcup (S)$
	$(2V_1^1) \sqcup (V_1^1 \sqcup S)$	$(V_1^{-1} \sqcup S) \sqcup (\varnothing)$
	$(V_1^1 \sqcup V_1^{-1}) \sqcup (V_1^{-1} \sqcup S)$	$(V_1^{-1}) \sqcup (S)$
	$(2V_1^1) \sqcup (2V_1^1)$	$(2S) \sqcup (\varnothing)$
	$(V_1^1 \sqcup V_1^{-1}) \sqcup (V_1^1 \sqcup V_1^{-1})$	$(S) \sqcup (S)$

TABLE 5 (continued)

$E_{\mathbb{R}}$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$
(continued)	$(V_1^1 \sqcup V_1^{-1}) \sqcup (S)$ $(2V_1^1) \sqcup (S)$ $(V_1^1 \sqcup S) \sqcup (V_1^1)$ $(V_1^1 \sqcup V_1^{-1}) \sqcup (V_1^1)$ $(2V_1^{-1}) \sqcup (V_1^{-1})$	$(V_1^1 \sqcup V_1^{-1}) \sqcup (S)$ $(V_1^{-1} \sqcup S) \sqcup (V_1^1)$ $(V_1^1 \sqcup S) \sqcup (V_1^1)$ $(V_1^1 \sqcup S) \sqcup (S)$ $(V_1^1) \sqcup (2S)$
$2V_1 \sqcup 3S$	$(V_1^1 \sqcup V_1^{-1} \sqcup S) \sqcup (2S)$ $(V_1^1 \sqcup S) \sqcup (2S)$ $(V_1^1 \sqcup S) \sqcup (V_1^1 \sqcup S)$ $(V_1^{-1} \sqcup S) \sqcup (S)$ $(V_1^1 \sqcup V_1^{-1}) \sqcup (S)$ $(V_1^1) \sqcup (V_1^1)$	$\varnothing$ $(V_1^{-1}) \sqcup (\varnothing)$ $(S) \sqcup (\varnothing)$ $(V_1^1) \sqcup (S)$ $(S) \sqcup (S)$ $(2S) \sqcup (S)$
$S_1 \sqcup 4S$	$(S_1^0) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$
$4S$	$(2S) \sqcup (2S)$ $(S) \sqcup (S)$	$\varnothing$ $(S) \sqcup (S)$
$V_6 \sqcup 2S$	$(V_6^4 \sqcup S) \sqcup (S)$ $(V_6^4) \sqcup (\varnothing)$	$\varnothing$ $(S) \sqcup (S)$
$V_5 \sqcup V_1 \sqcup S$	$(V_5^3 \sqcup V_1^1) \sqcup (S)$ $(V_5^{-3}) \sqcup (S)$ $(V_5^{-3}) \sqcup (V_1^1)$ $(V_5^3) \sqcup (\varnothing)$	$\varnothing$ $(V_1^{-1}) \sqcup (\varnothing)$ $(S) \sqcup (\varnothing)$ $(V_1^1) \sqcup (S)$
$V_4 \sqcup V_2 \sqcup S$	$(V_4^4) \sqcup (S)$ $(V_4^2) \sqcup (\varnothing)$	$(V_2^0) \sqcup (\varnothing)$ $(V_2^2) \sqcup (S)$
$V_4 \sqcup 2V_1$	$(V_4^4) \sqcup (V_1^1 \sqcup V_1^{-1})$ $(V_4^4) \sqcup (V_1^1)$ $(V_4^4) \sqcup (\varnothing)$	$\varnothing$ $(V_1^1) \sqcup (\varnothing)$ $(V_1^1) \sqcup (V_1^1)$
$V_4 \sqcup S_1 \sqcup S$	$(V_4^4) \sqcup (S)$	$(S_1^0) \sqcup (\varnothing)$
$V_4 \sqcup S$	$(V_4^4) \sqcup (S)$	$\varnothing$
$2V_3 \sqcup S$	$(V_3^3) \sqcup (S)$	$(V_3^1) \sqcup (\varnothing)$
$V_3 \sqcup V_2 \sqcup V_1$	$(V_3^3) \sqcup (V_1^1)$ $(V_3^3) \sqcup (\varnothing)$	$(V_2^2) \sqcup (\varnothing)$ $(V_2^2) \sqcup (V_1^1)$
$2V_2 \sqcup S_1$	$(V_2^2) \sqcup (V_2^2)$	$(S_1^4) \sqcup (\varnothing)$
$V_{11} \sqcup V_1$	$(V_{11}^1 \sqcup V_1^{-1}) \sqcup (\varnothing)$ $(V_{11}^1) \sqcup (\varnothing)$	$\varnothing$ $(V_1^{-1}) \sqcup (\varnothing)$
$V_{10} \sqcup V_2$	$(V_{10}^0) \sqcup (\varnothing)$	$(V_2^0) \sqcup (\varnothing)$
$V_{10} \sqcup S_1$	$(V_{10}^0) \sqcup (\varnothing)$	$(S_1^0) \sqcup (\varnothing)$

TABLE 5 (continued)

$E_{\mathbb{R}}$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$
V_{10}	$(V_{10}^{0}) \sqcup (\varnothing)$	$\varnothing$
$V_9 \sqcup V_3$	$(V_9^{-1}) \sqcup (\varnothing)$	$(V_3^{1}) \sqcup (\varnothing)$
$V_8 \sqcup V_4$	$(V_8^{-2}) \sqcup (\varnothing)$	$(V_4^{2}) \sqcup (\varnothing)$
$V_7 \sqcup V_5$	$(V_7^{-3}) \sqcup (\varnothing)$	$(V_5^{3}) \sqcup (\varnothing)$
$2V_6$	$(V_6^{4}) \sqcup (\varnothing)$	$(V_6^{4}) \sqcup (\varnothing)$