Introduction

Enriques surfaces play a special rôle in the theory of surfaces, both algebraic and analytic. They form a separate class in the Enriques-Kodaira classification of minimal surfaces: it is one of the four classes of Kodaira dimension 0 (the three others are abelian, hyperelliptic, and $K3$-surfaces). For an algebraist, a particular feature of Enriques surfaces is that they are irrational and have no holomorphic differential forms. For a topologist, they are the simplest examples of smooth 4-manifolds with even intersection form and signature not divisible by 16.

The fundamental group of an Enriques surface is $\mathbb{Z}_2$. Its universal covering is a $K3$-surface and, *vice versa*, the orbit space of a fixed point free holomorphic involution on a $K3$-surface is an Enriques surface. To visualize an Enriques surface one can pick a nonsingular curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4,4)$ invariant with respect to $s \times s$, where $s\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a holomorphic involution; if $s \times s$ has no fixed points in D, it lifts to a fixed point free involution of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over D; the orbit space of such a lift is an Enriques surface.

Over the field of complex numbers there is no difference between algebraic and analytic Enriques surfaces. Thus, a real Enriques surface can equally be regarded as either a complex analytic Enriques surface equipped with an anti-holomorphic involution or an algebraic Enriques surface defined over $\mathbb{R}$. The complex Enriques surfaces constitute a single deformation family; in particular, they are all diffeomorphic to each other. On the contrary, over $\mathbb{R}$ there are more than 200 of distinct deformation types. Even the topology of the real part of a surface can vary substantially.

Apart from the surfaces of general type (which are still not well understood even from the complex point of view), Enriques surfaces are one of the few cases left where the deformation classification of real structures was not known. Recall that real abelian surfaces, as well as minimal real rational surfaces, were essentially classified by A. Comessatti (see [Si] and [CP] for references), and real $K3$-surfaces, by V. Kharlamov and V. Nikulin (see [Kh4], [Kh5], and mainly [N1]). The case of hyperelliptic surfaces (the remaining class of Kodaira dimension 0) is still in progress and should be completed soon (see [CF]).

The topological study of real Enriques surfaces was started by V. Nikulin [N8] (see also some examples and references in [Si]). The topological classification of the real parts was completed by A. Degtyarev and V. Kharlamov [DK1], who also gave in [DK2] a more refined classification of the so called *half decompositions.* (The real part $E_{\mathbb{R}}$ of a real Enriques surface E splits in a natural way into two halves: two components of $E_{\mathbb{R}}$ are in the same half if a loop composed of two conjugate paths joining the components represents the nontrivial element of $\pi_1(E) = \mathbb{Z}_2$, see 9.2 for details. The half decomposition is a deformation invariant of the surface.) Both the classifications are finite. Apparently, this should reflect a more general statement: given a differential type M of 4-manifolds, the number of deformation classes of real surfaces E with E diffeomorphic to M is finite. (The proof would consist of two parts: first, the number of nonisomorphic real forms of a fixed complex surface is finite, *cf.* D.1, and, second, the map from the moduli space of real surfaces to the real part of the moduli space of complex surfaces is nice in some appropriate sense. Note that the number of complex deformation classes within M is known to be finite, see [FM].)

The principal purpose of this book is a systematic study of real Enriques surfaces, which culminates in their classification up to deformation. It turns out that *the deformation class of a real Enriques surface is determined by the topology of its complex conjugation involution*. This result, which is not obvious *à priori*, probably does not generalize to all other classes of surfaces. Note though, that a similar statement does hold for rational, abelian, and $K3$-surfaces, *i.e.*, in all the cases where the classification is completed. (A class of varieties for which the deformation type of a real structure is determined by its topology is called *quasi simple.*) Furthermore, we indicate simple explicit topological invariants sufficient for identifying the deformation classes of real Enriques surfaces. Some of the invariants are new and can be applied to other classes of surfaces or higher-dimensional varieties. This is one of the multiple by-products of our research.

The resulting lists of the topological types of the real part and half decompositions realized by real Enriques surfaces are collected in Summary, where one can also find extensive references to definitions and proofs. A complete list of deformation classes is found in Appendix C. In fact, the classification is slightly more ramified: we study real Enriques surfaces with a distinguished half of $E_{\mathbb{R}}$. Two classes of surfaces, namely, those with $E_{\mathbb{R}} = \{2V_1 \sqcup S\} \sqcup \{2V_1 \sqcup S\}$ and $E_{\mathbb{R}} = \{S_1\} \sqcup \{S_1\}$ have homeomorphic halves which cannot be interchanged by a deformation.

Note that a considerable part of our study is of purely topological nature. Thus, most prohibitions on the topology of the real part and on the half decomposition do not rely on the complex structure and hold, in fact, for so called *flexible real Enriques surfaces*, *i.e.*, smooth 4-manifolds with involution which mimic a few topological properties of real Enriques surfaces. Some results extend to *generalized real Enriques surfaces* (see 9.1) and reflect some new phenomena specific for real surfaces with not simply connected complexification (the case seldom considered in the literature). We develop several general tools related to topology of involutions, such as *Kalinin's spectral sequence* (with both $\mathbb{Z}_2$- and twisted $\mathbb{Z}$-coefficients) and *Viro homomorphisms*, and study their multiplicative properties and relation to the Smith exact sequence.

To attack the classification problem we use two different approaches. We divide the real Enriques surfaces into three types, *hyperbolic*, *parabolic*, and *elliptic*, according to whether the minimal Euler characteristic of the components of the real part is negative, zero, or positive, respectively. (This terminology reflects another, more geometric point of view, which is probably better adapted to higher dimensions: it seems to be crucial whether there is a component of the real part admitting a hyperbolic or flat metric, *cf.* [Vit]). The first approach converts real Enriques surfaces to real rational surfaces of a special kind, so called *DPN-surfaces*, which are characterized by the property that the anti-bicanonical system contains a nonsingular curve. We study (pluri-)canonical models of real *DPN*-surfaces and give their deformation classification in the case when the anti-bicanonical curve has a component of positive genus (*i.e.*, the corresponding real Enriques surface is of hyperbolic or parabolic type). To cover the case of real Enriques surfaces of elliptic type we use an alternative approach, based on an explicit description of the period spaces and a meticulous study of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions in the intersection lattice of a $K3$-surface. Note that the second approach is more universal, as it also applies to the hyperbolic and parabolic cases. However, the calculations involved are somewhat tedious (although they have been done), and, on the other hand, we do like the clarity and straightforwardness of the geometric construction. In both the approaches the underlying ideas are quite simple; the proofs use the twistor families and surjectivity of the period map.

The first, geometric approach reduces the task to the study of certain auxiliary objects, often classical, *e.g.*, real plane quartics, space cubics, intersections of two quadrics in $\mathbb{P}^4_{\mathbb{R}}$, and so on. Thus, one can switch the accents and go in the opposite direction, obtaining new proofs of classical results from the arithmetic calculations for real Enriques surfaces. To our surprise, for some of the auxiliary objects encountered in our study we could not find appropriate classification statements in the literature. We treat them in Appendix A, using traditional geometric methods; however, the results could as well be derived from the classification of real Enriques surfaces.

A few special classes of real Enriques surfaces are of particular interest. These are the maximal surfaces (in the sense of Smith theory, *i.e.*, those with the maximal total $\mathbb{Z}_2$-Betti number $\beta_*(E_{\mathbb{R}}) = 16$), surfaces with flat real part, and surfaces without real points. When an Enriques surface E is maximal (as well as in some other cases, see S5), one of the essential new topological invariants is the *Pontrjagin-Viro form.* It takes values in $\mathbb{Z}_4$ and is defined on certain (nonhomogeneous) elements

$\langle C_a - C_b\rangle + \alpha \in H_0(E_{\mathbb{R}};\mathbb{Z}_2)\oplus H_1(E_{\mathbb{R}};\mathbb{Z}_2)$ (where C_i are components of $E_{\mathbb{R}}$). The Pontrjagin-Viro form $\mathcal{P}$ defines, in particular, a decomposition of the real part into *quarters*: two components C_a, C_b are in the same half if and only if $\mathcal{P}$ is defined on $\langle C_a - C_b\rangle$, and they are said to be in same quarter if $\mathcal{P}(\langle C_a - C_b\rangle) = 0$. Following G. Mikhalkin [Mik] we call the decomposition into quarters *complex separation*.

Real Enriques surfaces without real points play a special rôle in the theory of Einstein manifolds. Recall that, due to N. Hitchin [Hi], the Euler characteristic $\chi(E)$ and signature $\sigma(E)$ of a compact orientable 4-dimensional Einstein manifold E satisfy the inequality $|\sigma(E)| \leqslant \frac{2}{3}\chi(E)$, the equality holding if and only if either E is flat or the universal covering X of E is a $K3$-surface and $\pi_1(E) = 1$, $\mathbb{Z}_2$, or $\mathbb{Z}_2\times\mathbb{Z}_2$. In the latter case, E is a $K3$-surface if $\pi_1 = 1$, an Enriques surface if $\pi_1 = \mathbb{Z}_2$, or the quotient of an Enriques surface by a free antiholomorphic involution if $\pi_1 = \mathbb{Z}_2\times\mathbb{Z}_2$. We call the Einstein manifolds of the last type *Enriques-Einstein-Hitchin varieties*. The manifolds of the other three extremal types (flat, $\pi_1 = 1$, and $\pi_1 = \mathbb{Z}_2$) are known to form connected families: two varieties with isomorphic fundamental groups can be continuously deformed into each other. Apparently, the number of connected components of the moduli space of Enriques-Einstein-Hitchin varieties was not known. The answer is given in [DK4], where it is proved that *the moduli space of Enriques-Einstein-Hitchin varieties is connected.* The proof is based on the Calabi-Yau theorem, which gives means for constructing a homotopy equivalence between the moduli space of Enriques-Einstein-Hitchin varieties and that of Enriques surfaces with free anti-holomorphic involution (*cf.* [Ito]), and on a thorough study of real elliptic pencils. In particular, in [DK4] explicit models of all empty real Enriques surfaces are obtained. In this book we give a different proof, based on the arithmetic approach used for the classification of the surfaces of elliptic type in general.

Real Enriques surfaces with flat real part might be used for constructing exotic irreducible 4-manifolds via a double covering ramified in a real torus, followed by Kodaira surgery (*cf.* [Sz]).

Plan of the book. The book consists of two parts. In the first one we prepare the necessary tools. As a consequence, the general theory is intermixed with a number of very specific technical statements, which are used explicitly in the further proofs. Such statements can be skipped at the first reading.

Chapter I deals with topology of involutions. We remind known facts (the Smith exact sequence, transfer map, and Lefschetz fixed point formula) and introduce new tools, such as Kalinin's spectral sequence and Viro homomorphisms. We pay special attention to involutions on manifolds and study in details the relation between Kalinin's spectral sequence and Poincaré duality. This gives rise to so called *Kalinin's intersection pairing* which plays a crucial rôle throughout the book.

Chapter II is devoted to arithmetic of integer lattices and finite quadratic forms. Our main subject here is finite quadratic forms on groups of period 4, which appear as the discriminant forms of the eigensublattices of $(\mathbb{Z}_2\times\mathbb{Z}_2)$-actions on unimodular lattices. The results obtained in this chapter are used in calculations in Chapter VIII, as well as in the proofs of some general prohibitions on the topology of generalized flexible real Enriques surfaces, see Chapter V.

In Chapter III we remind basic facts on algebraic surfaces, divisors, linear systems, singularities, and real structures. Furthermore, we introduce the notion of *DPN*-surfaces and study their properties. In particular, we give a detailed description of degree 1 and 2 maps of *DPN*-surfaces and singularities of anti-bicanonical curves and use it to construct certain natural models of *DPN*-pairs. We end the chapter with a brief discussion of deformations of curves and surfaces and prove a few technical statements on deformations of *DPN*-pairs. The results are mainly used in Chapter VII, where we carry a geometric study of real Enriques surfaces of hyperbolic and parabolic types.

Chapter IV deals with the topological aspects of theory of real algebraic surfaces and, more generally, real algebraic varieties. We cite the classical restrictions (Gudkov type congruences and Petrovsky type inequalities) and give a brief account of the deformation classification of real $K3$-surfaces. Besides, we specialize to real algebraic surfaces the results of Chapter I and study the conditions necessary for a double branched covering of a real algebraic surface to have empty real part.

In Part II we apply the auxiliary material of Part I to prove the main result of the book, the classification of real Enriques surfaces up to deformation. We start Part II with a summary, which contains the statements of the classification theorems and a brief guide to their proofs.

Chapter V is mainly topological. We use the results of Chapters I and II and study in details the homology of a generalized real Enriques surface and its covering generalized real $K3$-surface. As a consequence, we obtain certain prohibitions on the topology of the real structure. When specified to true real Enriques surfaces, this gives a complete set of restrictions on the topology of the real part, half decomposition, and Pontrjagin-Viro form. (The completeness, *i.e.*, existence of appropriate surfaces, is proved in Chapters VII and VIII.) Chapter V contains also some preliminary material used further in calculations.

Chapter VI deals with various period spaces related to $K3$-surfaces and Klein actions on them. We prove that the deformation class of a real Enriques surface is determined by its homological type, *i.e.*, the isomorphism class of the induced $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on the intersection lattice of the covering $K3$-surface. This result is crucial in Chapter VIII, where such actions are classified. Furthermore, it implies that a real Enriques surface is determined up to deformation by the topology of its real structure. We also describe here Donaldson's trick, which consists in changing the complex structure on a $K3$-surface so that an anti-holomorphic involution becomes holomorphic.

Chapters VII and VIII enumerate the deformation classes of real Enriques surfaces and, thus, complete the proof of the classification theorem. Chapter VII uses the geometric approach (based on Donaldson's trick), which works well for surfaces of hyperbolic and parabolic types. Chapter VIII contains the arithmetic calculations necessary to treat surfaces of elliptic type. Some surfaces of hyperbolic and parabolic types are reconsidered here in order to clarify the correspondence between deformation classes and homological types.

The book contains several appendices, where we collected everything that we did not find a better place for. Appendix A introduces the reader to the world of rigid isotopies of real algebraic varieties. We give a brief account of known facts, concerning planes curves, cubic surfaces, and intersections of quadrics. Besides, we state and prove a few other results, which we could not find in the literature. The most noteworthy among them is the rigid isotopy classification of plane quartics with double tangent.

Appendix B deals with a particular model of Enriques surfaces, the so called Horikawa construction. We show that, with few exceptions, this construction produces all real Enriques surfaces. At the end, we discuss some generalizations: first, we extend the Horikawa construction and show that it produces a number of generalized Enriques surfaces, and then we describe a particular construction of a Spin version of generalized Enriques surfaces.

Appendix C contains a complete list of deformation classes of real Enriques surfaces and corresponding homological types. It is organized as a determinant of real Enriques surfaces: we explain how to recover all the topological invariants considered in the book from the induced $(\mathbb{Z}_2\times\mathbb{Z}_2)$-action on the intersection lattice of the covering $K3$-surface.

In Appendix D we discuss various finiteness results on the number of real structures on a complex variety. As a by-product, we show that the kernel of the natural representation of the automorphism group of a $K3$-surface in its Picard group is cyclic. To our knowledge, this result is found in the literature only for algebraic $K3$-surfaces.

How to read this book. As usual, the end of a proof is marked with a $\square$. If $\square$ is placed at the end of a statement, this means that either the statement is trivial or its proof has already been explained. Some statements are marked with a $\rhd$, possibly followed by a reference. This means that the proof is not found in the book and the reader is directed to the literature. If most statements of a particular section are proved in the same book or paper, we give a single reference at the very beginning and do not repeat it after each statement.

Most notation introduced in the book are collected in the Glossary, where we also give a brief explanation of other more or less common terms and notation used throughout the book.

Acknowledgements. We are grateful to the following persons and institutions: S. Finashin for attracting our attention to Donaldson's trick converting a real structure on a $K3$-surface into a holomorphic involution, D. Harari for pointing out to [BS] and useful discussions, J. Lannes for communicating to us his unpublished proof (joint with S. Zarati) of a general formula relating Pontrjagin and Steenrod squares (see 2.5(4); our initial proof was for dimension 4 only), A. Marin for communicating to us an alternative approach to extending the Rokhlin-Guillou-Marin formula (see 2.6.1) and unpublished related results of C. Bailly, V. Nikulin for bringing to our attention Yu. Zarhin's paper [Za], our home institutions (Bilkent University, *Institut de Recherche Mathématique de Rennes*, and *Université Louis Pasteur*) for providing to us multiple opportunities to meet in person while working on this book, *Mathematische Forshungsinstitut* in Oberwolfach and its RiP program supported by *Volkswagen-Stiftung* for the hospitality and the excellent conditions which helped us to launch the project, *Dipartimento di Matematica* of *Università di Pisa* for the hospitality and an opportunity to test the principal parts

of this book in a graduate course, and the gratifying audience of the course delivered in Pisa for its patience. We are also thankful to the audience of our various talks on real Enriques surfaces and related subjects for the instructive questions and encouragement.