APPENDIX C

Determination of real Enriques surfaces

In this appendix, in Table 16, we give a list of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions on $H_2(X;\mathbb{Z}) = 2E_8 \oplus 3U$ resulting from real Enriques surfaces and corresponding deformation classes of the surfaces. The actions are listed according to their numeric type $(r^+, p^{(1)}, p^{(2)}, \delta, \epsilon, \delta^{(1)}, \delta^{(2)}, \epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{+-}, \epsilon^{-+}, \epsilon_0)$ (see 21.3.7), with the value of ϵ_0 ignored. The table is divided into three parts, corresponding to cases (1), (2), and (3) of 10.1.6 (which, in turn, correspond to $(\epsilon^{+-}, \epsilon^{-+}) = (1,1)$, $(0,0)$, and $(1,0)$, respectively). Each part is further subdivided into groups according to (r^+, δ, ϵ). (Recall that $r^+ = \operatorname{rk} L^{++} = 4 + \frac{1}{2}\chi(E_{\mathbb{R}})$, $\delta = \operatorname{lng}\Gamma$, and $\epsilon = 0$ or 1 if $(\Gamma, \circ)$ is even or, respectively, odd.) Finally, within each group the actions are sorted according to the value of r^{+-}. (We use $r^{+-} = \operatorname{rk} L^{+-} = r^- + p^{(1)}$ and $r^{-+} = \operatorname{rk} L^{-+} = r^- + p^{(2)}$ instead of $p^{(1)}$ and $p^{(2)}$, respectively.) The pairs $(r^{+-}, \delta^{(1)})$ and $(r^{-+}, \delta^{(2)})$ are listed in the table; if $\epsilon^{(i)} = 0$, i.e, the discriminant group is even, the corresponding pair is marked with an asterisk *. In order to avoid repetitions caused by interchanging the halves, in cases (1) and (2) we assume $r^{+-} \geqslant r^{-+}$. **Dg:** intro written

For each numeric type we indicate the topology of the halves $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ and their types (I_0, I_u, or II). If a numeric type corresponds to several actions and, hence, several deformation families, we give a reference to comments below.

All results are a direct consequence of Chapter VIII (more detailed references are given in the comments) and a tedious enumeration of numeric types satisfying 21.3.8 (which was mainly done with Maple).

TABLE 16. Determination of real Enriques surfaces

Case 10.1.6(1): $\mathcal{L}^{+-}$, $\mathcal{L}^{-+}$ are odd

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=0$, $\delta=0$,	$(\Gamma,\circ)$ is even;	all surfaces are of type I_u		
$(11,11)$	$(1,1)$	V_1	V_{11}	
$(10,10)$	$(2,2)$	V_2	V_{10}	
$(9,9)$	$(3,3)$	V_3	V_9	
$(8,8)$	$(4,4)$	V_4	V_8	
$(7,7)$	$(5,5)$	V_5	V_7	
$(6,6)$	$(6,6)$	V_6	V_6	
$r^+=1$, $\delta=1$,	$(\Gamma,\circ)$ is odd			
$(10,11)$	$(2,3)$	V_1	V_9	
$(9,10)$	$(3,4)$	V_2	V_8	
$(8,9)$	$(4,5)$	V_3	V_7	
$(7,8)$	$(5,6)$	V_4	V_6	
$(6,7)$	$(6,7)$	V_5	V_5	
$r^+=2$, $\delta=2$,	$(\Gamma,\circ)$ is odd			
$(9,11)$	$(3,5)$	V_1	V_7	
$(8,10)$	$(4,6)$	V_2	V_6	
$(7,9)$	$(5,7)$	V_3	V_5	
$(6,8)$	$(6,8)$	V_4	V_4	
$r^+=3$, $\delta=3$,	$(\Gamma,\circ)$ is odd			
$(8,11)$	$(4,7)$	V_1	V_5	
$(7,10)$	$(5,8)$	V_2	V_4	
$(6,9)$	$(6,9)$	V_3	V_3	
$r^+=4$, $\delta=2$,	$(\Gamma,\circ)$ is even;	all surfaces are of type I_u		
$(9,9)$	$(3,7)$	$V_1\sqcup S$	V_5	
$(8,10)$	$(4,8)$	$2V_1$	V_4	
$(8,10)$	$(4,6)$	S	$V_5\sqcup V_1$	
$(8,8)$	$(4,8)$	$V_2\sqcup S$	V_4	
$(7,11)$	$(5,7)$	V_1	$V_4\sqcup V_1$	
$(7,11)$	$(5,5)$	V_1	$V_5\sqcup S$	
$(7,9)$	$(5,9)$	$V_2\sqcup V_1$	V_3	
$(7,7)$	$(5,9)$	$V_3\sqcup S$	V_3	
$(6,10)$	$(6,8)$	V_2	$V_3\sqcup V_1$	
$(6,10)$	$(6,6)$	V_2	$V_4\sqcup S$	
$r^+=4$, $\delta=4$,	$(\Gamma,\circ)$ is odd			
$(7,11)$	$(5,9)$	V_1	V_3	
$(6,10)$	$(6,10)$	V_2	V_2	

Table 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=5,\quad \delta=3,\quad (\Gamma,\circ)$ is odd				
$(8,9)$	$(4,9)$	$V_1\sqcup S$	V_3	
$(7,10)$	$(5,10)$	$2V_1$	V_2	
$(7,10)$	$(5,8)$	S	$V_3\sqcup V_1$	
$(7,8)$	$(5,10)$	$V_2\sqcup S$	V_2	
$(6,11)$	$(6,9)$	V_1	$V_2\sqcup V_1$	
$(6,11)$	$(6,7)$	V_1	$V_3\sqcup S$	
$r^+=5,\quad \delta=5,\quad (\Gamma,\circ)$ is odd				
$(6,11)$	$(6,11)$	V_1	V_1	
$r^+=6,\quad \delta=2,\quad (\Gamma,\circ)$ is odd				
$(9,7)$	$(3,9)$	$V_1\sqcup 2S$	V_3	
$(8,8)$	$(4,10)$	$2V_1\sqcup S$	V_2	
$(8,8)$	$(4,8)$	$2S$	$V_3\sqcup V_1$	
$(8,6)$	$(4,10)$	$V_2\sqcup 2S$	V_2	
$(7,9)$	$(5,11)$	$3V_1$	V_1	
$(7,9)$	$(5,9)$	$V_1\sqcup S$	$V_2\sqcup V_1$	
$(7,9)$	$(5,7)$	$V_1\sqcup S$	$V_3\sqcup S$	
$(7,7)$	$(5,11)$	$V_2\sqcup V_1\sqcup S$	V_1	
$(7,5)$	$(5,11)$	$V_3\sqcup 2S$	V_1	
$(6,10)$	$(6,10)$	$2V_1$	$2V_1$	
$(6,10)$	$(6,8)$	$2V_1$	$V_2\sqcup S$	see 1°
		S	$V_2\sqcup 2V_1$	see 1°
$(6,10)$	$(6,6)$	S	$V_3\sqcup V_1\sqcup S$	
$(6,8)$	$(6,8)$	$V_2\sqcup S$	$V_2\sqcup S$	
$r^+=6,\quad \delta=4,\quad (\Gamma,\circ)$ is odd				
$(7,9)$	$(5,11)$	$V_1\sqcup S$	V_1	
$(6,10)$	$(6,10)$	$2V_1$	S	see 3°
		S	$2V_1$	see 3°
$r^+=7,\quad \delta=1,\quad (\Gamma,\circ)$ is odd				
$(10,5)$	$(2,9)$	$V_1\sqcup 3S$	V_3	
$(9,6)$	$(3,10)$	$2V_1\sqcup 2S$	V_2	
$(9,6)$	$(3,8)$	$3S$	$V_3\sqcup V_1$	
$(9,4)$	$(3,10)$	$V_2\sqcup 3S$	V_2	
$(8,7)$	$(4,11)$	$3V_1\sqcup S$	V_1	
$(8,7)$	$(4,9)$	$V_1\sqcup 2S$	$V_2\sqcup V_1$	
$(8,7)$	$(4,7)$	$V_1\sqcup 2S$	$V_3\sqcup S$	
$(8,5)$	$(4,11)$	$V_2\sqcup V_1\sqcup 2S$	V_1	
$(8,3)$	$(4,11)$	$V_3\sqcup 3S$	V_1	

TABLE 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=7$, $\delta=1$, $(\Gamma,\circ)$ is odd				(continued)
(7,8)	(5,10)	$4V_1$	S	see 1°
		$2V_1\sqcup S$	$2V_1$	see 1°
(7,8)	(5,8)	$2V_1\sqcup S$	$V_2\sqcup S$	see 1°
		$2S$	$V_2\sqcup 2V_1$	see 1°
(7,8)	(5,6)	$2S$	$V_3\sqcup V_1\sqcup S$	
(7,6)	(5,10)	$V_2\sqcup 2V_1\sqcup S$	S	see 1°
		$V_2\sqcup 2S$	$2V_1$	see 1°
(7,6)	(5,8)	$V_2\sqcup 2S$	$V_2\sqcup S$	
(7,4)	(5,10)	$V_3\sqcup V_1\sqcup 2S$	S	
(6,9)	(6,9)	$3V_1$	$V_1\sqcup S$	see 1°
		$V_1\sqcup S$	$3V_1$	see 1°
(6,9)	(6,7)	$V_1\sqcup S$	$V_2\sqcup V_1\sqcup S$	
(6,9)	(6,5)	$V_1\sqcup S$	$V_3\sqcup 2S$	
$r^+=7$, $\delta=3$, $(\Gamma,\circ)$ is odd				
(8,7)	(4,11)	$V_1\sqcup 2S$	V_1	
(7,8)	(5,10)	$2V_1\sqcup S$	S	see 3°
		$2S$	$2V_1$	see 3°
(6,9)	(6,9)	$V_1\sqcup S$	$V_1\sqcup S$	
$r^+=8$, $\delta=0$, $(\Gamma,\circ)$ is even; all surfaces are of type I_u				
(11,3)	(1,9)	$V_1\sqcup 4S$	V_3	
(10,4)	(2,10)	$2V_1\sqcup 3S$	V_2	2 classes, see 2°
(10,4)	(2,8)	$4S$	$V_3\sqcup V_1$	
(10,2)	(2,10)	$V_2\sqcup 4S$	V_2	see 2°
(9,5)	(3,11)	$3V_1\sqcup 2S$	V_1	2 classes, see 2°
(9,5)	(3,9)	$V_1\sqcup 3S$	$V_2\sqcup V_1$	4 classes, see 2°
(9,5)	(3,7)	$V_1\sqcup 3S$	$V_3\sqcup S$	
(9,3)	(3,11)	$V_2\sqcup V_1\sqcup 3S$	V_1	3 classes, see 2°
(9,1)	(3,11)	$V_3\sqcup 4S$	V_1	
(8,6)	(4,10)	$4V_1\sqcup S$	S	see 1°, 2°
		$2V_1\sqcup 2S$	$2V_1$	5 classes, see 1°, 2°
(8,6)	(4,8)	$2V_1\sqcup 2S$	$V_2\sqcup S$	4 classes, see 1°, 2°
		$3S$	$V_2\sqcup 2V_1$	2 classes, see 1°, 2°
(8,6)	(4,6)	$3S$	$V_3\sqcup V_1\sqcup S$	
(8,4)	(4,10)	$V_2\sqcup 2V_1\sqcup 2S$	S	2 classes, see 1°, 2°
		$V_2\sqcup 3S$	$2V_1$	3 classes, see 1°, 2°
(8,4)	(4,8)	$V_2\sqcup 3S$	$V_2\sqcup S$	2 classes, see 2°
(8,2)	(4,10)	$V_3\sqcup V_1\sqcup 3S$	S	
(7,7)	(5,9)	$3V_1\sqcup S$	$V_1\sqcup S$	3 classes, see 1°, 2°
		$V_1\sqcup 2S$	$3V_1$	2 classes, see 1°, 2°
(7,7)	(5,7)	$V_1\sqcup 2S$	$V_2\sqcup V_1\sqcup S$	4 classes, see 2°

Table 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=8$, $\delta=0$, $(\Gamma,\circ)$ is even				(continued)
$(7,7)$	$(5,5)$	$V_1\sqcup 2S$	$V_3\sqcup 2S$	
$(7,5)$	$(5,9)$	$V_2\sqcup V_1\sqcup 2S$	$V_1\sqcup S$	4 classes, see 2°
$(7,3)$	$(5,9)$	$V_3\sqcup 3S$	$V_1\sqcup S$	
$(6,8)$	$(6,8)$	$4V_1$	$2S$	2 classes, see 1°, 2°
		$2V_1\sqcup S$	$2V_1\sqcup S$	4 classes, see 1°, 2°
		$2S$	$4V_1$	2 classes, see 1°, 2°
$(6,8)$	$(6,6)$	$2V_1\sqcup S$	$V_2\sqcup 2S$	2 classes, see 1°, 2°
		$2S$	$V_2\sqcup 2V_1\sqcup S$	4 classes, see 1°, 2°
$(6,8)$	$(6,4)$	$2S$	$V_3\sqcup V_1\sqcup 2S$	
$(6,6)$	$(6,6)$	$V_2\sqcup 2S$	$V_2\sqcup 2S$	see 2°
$r^+=8$, $\delta=2$, $(\Gamma,\circ)$ is even; all surfaces are of type I_u				
$(9,5)$	$(3,11)$	$V_1\sqcup 3S$	V_1	
$(8,6)$	$(4,10)$	$2V_1\sqcup 2S$	S	see 3°
		$3S$	$2V_1$	see 3°
$(7,7)$	$(5,9)$	$V_1\sqcup 2S$	$V_1\sqcup S$	
$(6,8)$	$(6,8)$	$2V_1\sqcup S$	$2S$	see 3°
		$2S$	$2V_1\sqcup S$	see 3°
$r^+=8$, $\delta=2$, $(\Gamma,\circ)$ is odd				
$(9,5)$	$(3,11)$	$V_1\sqcup 3S$	V_1	
$(8,6)$	$(4,10)$	$2V_1\sqcup 2S$	S	see 3°
		$3S$	$2V_1$	see 3°
$(7,7)$	$(5,9)$	$V_1\sqcup 2S$	$V_1\sqcup S$	
$(6,8)$	$(6,8)$	$2V_1\sqcup S$	$2S$	see 3°
		$2S$	$2V_1\sqcup S$	see 3°
$r^+=9$, $\delta=1$, $(\Gamma,\circ)$ is odd				
$(10,3)$	$(2,11)$	$V_1\sqcup 4S$	V_1	see 4°
$(9,4)$	$(3,10)$	$2V_1\sqcup 3S$	S	see 3°, 4°
		$4S$	$2V_1$	see 3°, 4°
$(8,5)$	$(4,9)$	$V_1\sqcup 3S$	$V_1\sqcup S$	2 classes, see 4°
$(7,6)$	$(5,8)$	$2V_1\sqcup 2S$	$2S$	2 classes, see 3°, 4°
		$3S$	$2V_1\sqcup S$	see 3°, 4°
$(6,7)$	$(6,7)$	$V_1\sqcup 2S$	$V_1\sqcup 2S$	2 classes, see 4°

Case 10.1.6(2): $\mathcal{L}^{+-}$, $\mathcal{L}^{-+}$ are both even of full rank

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=0$, $\delta=0$, $(\Gamma,\circ)$ is even				
$(10,10)^*$	$(2,2)^*$	$\varnothing$	V_{10}	2 classes, see 5°

TABLE 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=1,\ \delta=1,\ (\Gamma,\circ)$ is odd				
$(9,10)^*$	$(3,4)$	$\varnothing$	V_8	type $(\mathrm{I}_0,\mathrm{II})$
$r^+=2,\ \delta=2,\ (\Gamma,\circ)$ is odd				
$(8,10)^*$	$(4,6)$	$\varnothing$	V_6	type $(\mathrm{I}_0,\mathrm{II})$
$r^+=3,\ \delta=3,\ (\Gamma,\circ)$ is odd				
$(7,10)^*$	$(5,8)$	$\varnothing$	V_4	type $(\mathrm{I}_0,\mathrm{II})$
$r^+=4,\ \delta=2,\ (\Gamma,\circ)$ is even				
$(6,10)^*$	$(6,10)^*$	$\varnothing$	$\varnothing$	type $(\mathrm{I}_0,\mathrm{I}_0)$
$(6,10)^*$	$(6,8)^*$	$\varnothing$	$2V_2$	type $(\mathrm{I}_0,\mathrm{I}_0)$, see 8°
		$\varnothing$	S_1	2 classes, see 8°
$(6,10)^*$	$(6,6)^*$	$\varnothing$	$V_4\sqcup S$	2 classes, see 5°
$(6,8)^*$	$(6,8)^*$	S_1	S_1	2 classes, see 9°
$r^+=4,\ \delta=4,\ (\Gamma,\circ)$ is odd				
$(6,10)^*$	$(6,10)$	$\varnothing$	V_2	type $(\mathrm{I}_0,\mathrm{II})$
$r^+=5,\ \delta=3,\ (\Gamma,\circ)$ is odd				
$(7,10)$	$(5,10)^*$	S	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(7,10)$	$(5,8)^*$	S	S_1	type $(\mathrm{II},\mathrm{I}_u)$
$(7,8)$	$(5,10)^*$	$V_2\sqcup S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=6,\ \delta=2,\ (\Gamma,\circ)$ is odd				
$(8,8)$	$(4,10)^*$	$2S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,8)$	$(4,8)^*$	$2S$	S_1	type $(\mathrm{II},\mathrm{I}_u)$
$(8,6)$	$(4,10)^*$	$V_2\sqcup 2S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=6,\ \delta=4,\ (\Gamma,\circ)$ is odd				
$(6,10)$	$(6,10)$	S	S	
$r^+=7,\ \delta=1,\ (\Gamma,\circ)$ is odd				
$(9,6)$	$(3,10)^*$	$3S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,6)$	$(3,8)^*$	$3S$	S_1	type $(\mathrm{II},\mathrm{I}_u)$
$(9,4)$	$(3,10)^*$	$V_2\sqcup 3S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=7,\ \delta=3,\ (\Gamma,\circ)$ is odd				
$(7,8)$	$(5,10)$	$2S$	S	
$r^+=8,\ \delta=0,\ (\Gamma,\circ)$ is even				
$(10,4)^*$	$(2,10)^*$	$4S$	$\varnothing$	4 classes, see 6°
$(10,4)^*$	$(2,8)^*$	$4S$	S_1	2 classes, see 7°
$(10,2)^*$	$(2,10)^*$	$V_2\sqcup 4S$	$\varnothing$	2 classes, see 5°

TABLE 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=8,\quad \delta=2,\quad (\Gamma,\circ)$ is even				
$(6,8)$	$(6,8)$	$2S$	$2S$	2 classes, see 5°
$r^+=8,\quad \delta=2,\quad (\Gamma,\circ)$ is odd				
$(8,6)$	$(4,10)$	$3S$	S	
$r^+=9,\quad \delta=1,\quad (\Gamma,\circ)$ is odd				
$(9,4)^*$	$(3,10)$	$4S$	S	2 classes, see 7°

Case 10.1.6(3): $\mathcal{L}^{+-}$, $\mathcal{L}^{-+}$ are even, $\mathcal{L}^{+-}$ is not of full rank

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=0,\quad \delta=0,\quad (\Gamma,\circ)$ is even; all surfaces are of type I_u				
$(10,8)^*$	$(2,2)^*$	S_1	V_{10}	type $(\mathrm{I}_u,\mathrm{I}_0)$
$(2,0)^*$	$(10,10)^*$	$V_{11}\sqcup V_1$	$\varnothing$	type $(\mathrm{I}_u,\mathrm{I}_0)$
$r^+=1,\quad \delta=1,\quad (\Gamma,\circ)$ is odd				
$(11,10)$	$(1,2)^*$	S	V_{10}	type $(\mathrm{II},\mathrm{I}_0)$
$(9,8)^*$	$(3,4)$	S_1	V_8	type $(\mathrm{I}_u,\mathrm{II})$
$(3,2)$	$(9,10)^*$	$V_9\sqcup V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=2,\quad \delta=2,\quad (\Gamma,\circ)$ is odd				
$(10,10)$	$(2,4)$	S	V_8	
$(8,8)^*$	$(4,6)$	S_1	V_6	type $(\mathrm{I}_u,\mathrm{II})$
$(4,4)$	$(8,10)^*$	$V_7\sqcup V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=3,\quad \delta=3,\quad (\Gamma,\circ)$ is odd				
$(9,10)$	$(3,6)$	S	V_6	
$(7,8)^*$	$(5,8)$	S_1	V_4	type $(\mathrm{I}_u,\mathrm{II})$
$(5,6)$	$(7,10)^*$	$V_5\sqcup V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=4,\quad \delta=2,\quad (\Gamma,\circ)$ is even; all surfaces are of type I_u				
$(10,8)$	$(2,6)$	$2S$	V_6	
$(6,8)^*$	$(6,8)^*$	S_1	$2V_2$	type $(\mathrm{I}_u,\mathrm{I}_0)$
$(6,8)^*$	$(6,6)^*$	S_1	$V_4\sqcup S$	type $(\mathrm{I}_u,\mathrm{I}_0)$
$(6,6)^*$	$(6,10)^*$	$V_4\sqcup 2V_1$	$\varnothing$	type $(\mathrm{I}_u,\mathrm{I}_0)$
$(6,4)^*$	$(6,10)^*$	$V_5\sqcup V_1\sqcup S$	$\varnothing$	type $(\mathrm{I}_u,\mathrm{I}_0)$
$r^+=4,\quad \delta=4,\quad (\Gamma,\circ)$ is odd				
$(8,10)$	$(4,8)$	S	V_4	
$(6,8)^*$	$(6,10)$	S_1	V_2	type $(\mathrm{I}_u,\mathrm{II})$
$(6,8)$	$(6,10)^*$	$V_3\sqcup V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$r^+=5,\quad \delta=3,\quad (\Gamma,\circ)$ is odd				
$(9,8)$	$(3,8)$	$2S$	V_4	

TABLE 16 (continued)

$(r^{+-},\delta^{(1)})$	$(r^{-+},\delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+=5,\quad \delta=3,\quad (\Gamma,\circ)$ is odd				(continued)
$(7,10)$	$(5,8)^*$	S	$2V_2$	type $(\mathrm{II},\mathrm{I}_0)$
$(7,10)$	$(5,6)^*$	S	$V_4\sqcup S$	type $(\mathrm{II},\mathrm{I}_0)$
$(7,8)$	$(5,10)^*$	$V_2\sqcup 2V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(7,6)$	$(5,10)^*$	$V_3\sqcup V_1\sqcup S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(5,8)^*$	$(7,8)$	S_1	$V_2\sqcup S$	type $(\mathrm{I}_u,\mathrm{II})$
$r^+=5,\quad \delta=5,\quad (\Gamma,\circ)$ is odd				
$(7,10)$	$(5,10)^*$	$2V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(7,10)$	$(5,10)$	S	V_2	
$r^+=6,\quad \delta=2,\quad (\Gamma,\circ)$ is odd				
$(10,6)$	$(2,8)$	$3S$	V_4	
$(8,8)$	$(4,10)^*$	$4V_1$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,8)$	$(4,8)^*$	$2S$	$2V_2$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,8)$	$(4,6)^*$	$2S$	$V_4\sqcup S$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,6)$	$(4,10)^*$	$V_2\sqcup 2V_1\sqcup S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,4)$	$(4,10)^*$	$V_3\sqcup V_1\sqcup 2S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(4,8)^*$	$(8,6)$	S_1	$V_2\sqcup 2S$	type $(\mathrm{I}_u,\mathrm{II})$
$r^+=6,\quad \delta=4,\quad (\Gamma,\circ)$ is odd				
$(8,8)$	$(4,10)^*$	$2V_1\sqcup S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(8,8)$	$(4,10)$	$2S$	V_2	
$(6,10)$	$(6,8)$	S	$V_2\sqcup S$	
$r^+=7,\quad \delta=1,\quad (\Gamma,\circ)$ is odd				
$(11,4)^*$	$(1,8)$	$4S$	V_4	type $(\mathrm{I}_u,\mathrm{II})$
$(9,6)$	$(3,10)^*$	$4V_1\sqcup S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,6)$	$(3,8)^*$	$3S$	$2V_2$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,6)$	$(3,6)^*$	$3S$	$V_4\sqcup S$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,4)$	$(3,10)^*$	$V_2\sqcup 2V_1\sqcup 2S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,2)$	$(3,10)^*$	$V_3\sqcup V_1\sqcup 3S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(3,8)^*$	$(9,4)$	S_1	$V_2\sqcup 3S$	type $(\mathrm{I}_u,\mathrm{II})$
$r^+=7,\quad \delta=3,\quad (\Gamma,\circ)$ is odd				
$(9,6)$	$(3,10)^*$	$2V_1\sqcup 2S$	$\varnothing$	type $(\mathrm{II},\mathrm{I}_0)$
$(9,6)$	$(3,10)$	$3S$	V_2	
$(7,8)$	$(5,8)$	$2S$	$V_2\sqcup S$	
$(5,10)$	$(7,6)$	S	$V_2\sqcup 2S$	
$r^+=8,\quad \delta=0,\quad (\Gamma,\circ)$ is even; all surfaces are of type I_u				
$(10,4)^*$	$(2,10)^*$	$4V_1\sqcup 2S$	$\varnothing$	2 classes, see 2°
$(10,4)^*$	$(2,8)^*$	$4S$	$2V_2$	4 classes, see 2°
$(10,4)^*$	$(2,6)^*$	$4S$	$V_4\sqcup S$	type $(\mathrm{I}_u,\mathrm{I}_0)$

Table 16 (continued)

$(r^{+-}, \delta^{(1)})$	$(r^{-+}, \delta^{(2)})$	$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	Comments
$r^+ = 8,\quad \delta = 0,\quad (\Gamma, \circ)$ is even				(continued)
$(10,2)^*$	$(2,10)^*$	$V_2 \sqcup 2V_1 \sqcup 3S$	$\varnothing$	3 classes, see 2°
$(10,0)^*$	$(2,10)^*$	$V_3 \sqcup V_1 \sqcup 4S$	$\varnothing$	type $(\mathrm{I}_u, \mathrm{I}_0)$
$(2,8)^*$	$(10,2)^*$	S_1	$V_2 \sqcup 4S$	
$r^+ = 8,\quad \delta = 2,\quad (\Gamma, \circ)$ is even; all surfaces are of type I_u				
$(10,4)^*$	$(2,10)^*$	$2V_1 \sqcup 3S$	$\varnothing$	type $(\mathrm{I}_u, \mathrm{I}_0)$
$(10,4)$	$(2,10)$	$4S$	V_2	
$(6,8)$	$(6,6)$	$2S$	$V_2 \sqcup 2S$	
$r^+ = 8,\quad \delta = 2,\quad (\Gamma, \circ)$ is odd				
$(10,4)^*$	$(2,10)$	$4S$	V_2	
$(10,4)$	$(2,10)^*$	$2V_1 \sqcup 3S$	$\varnothing$	type $(\mathrm{II}, \mathrm{I}_0)$
$(8,6)$	$(4,8)$	$3S$	$V_2 \sqcup S$	
$(4,10)$	$(8,4)$	S	$V_2 \sqcup 3S$	
$r^+ = 9,\quad \delta = 1,\quad (\Gamma, \circ)$ is odd				
$(11,2)$	$(1,10)^*$	$2V_1 \sqcup 4S$	$\varnothing$	type $(\mathrm{II}, \mathrm{I}_0)$; see 4°
$(9,4)^*$	$(3,8)$	$4S$	$V_2 \sqcup S$	2 classes, see 7°
$(3,10)$	$(9,2)^*$	S	$V_2 \sqcup 4S$	type $(\mathrm{II}, \mathrm{I}_0)$

Comments to the table.

es.d+2 1°. Surfaces with $E_{\mathbb{R}} = 4V_1 \sqcup kS$ or $V_2 \sqcup 2V_1 \sqcup kS$. The decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ is determined by the ranks $\operatorname{rk} \Delta_-^{+-}$ and $\operatorname{rk} \Delta_-^{-+}$, see 22.3.3.

es.m-d+2 2°. The deformation classes of surfaces with $E_{\mathbb{R}} = 4V_1 \sqcup 2S$, $V_2 \sqcup 2V_1 \sqcup 3S$, or $2V_2 \sqcup 4S$ are distinguished by the complex separation and the value $\mathcal{P}(w_1)$ of the Pontrjagin-Viro form on the characteristic class w_1 of a V_2-component. Arithmetically the surfaces are distinguished by the coarse types of Δ_-^{+-}, Δ_-^{-+} and Γ_-^{+-}, Γ_-^{-+}, see 22.3.2, 22.3.3, 22.3.4, and Tables 8–10.

Remark. The number of deformation classes of surfaces with $E_{\mathbb{R}} = \{2V_1 \sqcup S\} \sqcup \{2V_1 \sqcup S\}$ is four if one of the halves is marked and three otherwise; the halves are asymmetric if the complex separation is $\{(2V_1) \sqcup (S)\} \sqcup \{(V_1 \sqcup S) \sqcup (V_1)\}$.

es.2V1 3°. Surfaces with $E_{\mathbb{R}} = 2V_1 \sqcup kS$. The decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ is determined by the Brown invariant $\operatorname{Br}^{\perp}(\mathcal{L}^{+-}, \Delta_0^{+-})$, if the surface is of type II, or by the class $\xi_+^{+-} \in \Gamma_+^{+-} \subset \mathcal{L}^{++}$, if the surface is of type I, see 22.4.2.

es.m2V1 4°. The deformation classes of surfaces with $E_{\mathbb{R}} = 2V_1 \sqcup 4S$ are distinguished by the integral complex separation, see 22.4.1. Arithmetically the latter is determined by the coarse type of Γ_0^{+-}, see Table 11.

es.type 5°. Surfaces with $E_{\mathbb{R}} = \{V_{10}\} \sqcup \{\varnothing\}$, $\{V_4 \sqcup S\} \sqcup \{\varnothing\}$, $\{V_2 \sqcup 4S\} \sqcup \{\varnothing\}$, and $\{2S\} \sqcup \{2S\}$ have two deformation classes each; they are distinguished by the type, which may be I_u or I_0. Arithmetically the type I_u or I_0 is determined by the coarse type of $\bar{\Omega}^{+-}/\Gamma = [0]$ or $[1]$, respectively, see 10.1.7.

es.4S 6°. The four deformation classes of surfaces with $E_{\mathbb{R}} = \{4S\} \sqcup \{\varnothing\}$ are distinguished by the types of $E_{\mathbb{R}}$ in E and $X/t^{(2)}$ (topologically the latter is also a real Enriques surface). The two types may independently be I_u or I_0; arithmetically they are distinguished by the coarse types of $\bar\Omega^{+-}$ and Γ_-^{-+}, see 22.10.

es.4S+ 7°. Surfaces with $E_{\mathbb{R}} = \{4S\} \sqcup \{S\}$, $\{4S\} \sqcup \{V_2 \sqcup S\}$, and $\{4S\} \sqcup \{S_1\}$ have two deformation classes each; they are distinguished by the type of $E_{\mathbb{R}}^{(1)}$ in $X/t^{(2)}$, which is a topological real rational surface. Arithmetically the surfaces are distinguished by the coarse type of Γ_0^{-+}, see 22.10. (The surfaces with $E_{\mathbb{R}} = \{4S\} \sqcup \{S_1\}$ are also distinguished by the type of $E_{\mathbb{R}}$ in E; it is the same as the type of $E_{\mathbb{R}}^{(1)}$ in $X/t^{(1)}$.)

es.2S1 8°. Surfaces with $E_{\mathbb{R}} = \{2V_2\} \sqcup \{\varnothing\}$ and $\{S_1\} \sqcup \{\varnothing\}$ are distinguished arithmetically by the coarse types of $\bar\Omega^{+-}/\Gamma$ and Γ_-^{-+}/Γ, see 22.8: for $E_{\mathbb{R}} = \{2V_2\}$ one has $\bar\Omega^{+-}/\Gamma = \Gamma_-^{-+}/\Gamma = [1]$, and for $E_{\mathbb{R}} = \{S_1\}$ the pair $(\bar\Omega^{+-}/\Gamma, \Gamma_-^{-+}/\Gamma)$ is either $([0],[1])$ (the surface is of type I_u) or $([1],[0])$ (the surface is of type I_0).

es.S1+S1 9°. Surfaces with $E_{\mathbb{R}} = 2S_1$. There are two deformation classes, distinguished by the integral Pontrjagin-Viro form, see 9.8.9, or, equivalently, by the linking coefficient form, see 11.4.3. Arithmetically the surfaces are distinguished by the coarse type of $\Gamma_-^{+-}/\Gamma = \mathcal{U}_2$ or $\mathcal{V}_2$, see 22.9. The two classes differ if one of the halves is marked; they are interchanged by permuting the halves.