Topology of Algebraic Curves
An Approach *via Dessins d'Enfants*

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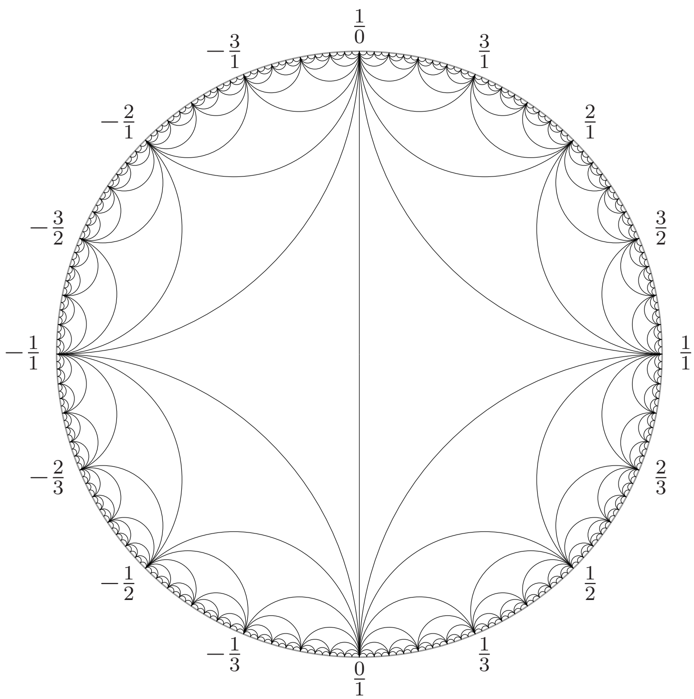

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Abstract: The book is an attempt to summarize and extend a number of results on the topology of trigonal curves in geometrically ruled surfaces. An emphasis is given to various applications of the theory to a few related areas, most notably singular plane curves of small degree, elliptic surfaces and Lefschetz fibrations (both complex and real), and Hurwitz equivalence of braid monodromy factorizations.

The approach relies on a close relation between trigonal curves/elliptic surfaces, a certain class of ribbon graphs, and subgroups of the modular group, which provides a combinatorial framework for the study of geometric objects. A brief summary of the necessary auxiliary results and techniques used and a background of the principal problems dealt with are included in the text.

The book is intended to researches and graduate students in the field of topology of complex and real algebraic varieties.

To Ayşe

Preface

The purpose of this monograph is to summarize, unify, and extend a number of interrelated results that were published or submitted during the last five years in a series of papers by the author, partially in collaboration with Ilia Itenberg, Viatcheslav Kharlamov, and Nermin Salepci. As often happens in a long research project, the principal ideas have evolved and a few minor mistakes have been discovered, calling for a new, more comprehensive and unified approach to the earlier papers. Furthermore, as the work is still in progress, I am representing older results in a more complete and general form. The monograph also contains several newer results that have never appeared elsewhere. Thus, we complete the analysis of the metabelian invariants of a trigonal curve (see chapter 6), compute the fundamental groups of all, not necessarily maximizing, irreducible simple sextics with a triple point (see chapter 8; a few new sextics with finite nonabelian fundamental group have been discovered), and establish the quasi-simplicity of most ribbon curves, including M-curves (see §10.3.2). The latest achievement is the complete understanding of simple monodromy factorizations of length two in the modular group (see §10.2.2; joint work with N. Salepci). In spite of its apparent simplicity, our description of such factorizations has interesting applications to the topology of real trigonal curves and real Lefschetz fibrations; they are discussed in §10.3. There also are a few other advances in the study of monodromy factorizations (see, *e.g.*, §10.2.3), but in general the situation still remains unclear and the problem seems wild.

The dominant theme of the book is the very fruitful close relation between three classes of objects:

- elliptic surfaces and trigonal curves in ruled surfaces, see chapter 3,
- skeletons (certain bipartite ribbon graphs), see chapter 1, and
- subgroups of the modular group $\Gamma := PSL(2, \mathbb{Z})$, see chapter 2.

(Slightly different versions of skeletons appeared in the literature under a number of names, the most well known being *dessins d'enfants* and quilts.) When restricted to appropriate subclasses, this relation becomes bijective, providing an intuitive combinatorial and topological framework for the study of trigonal curves, on the one hand, and of subgroups of Γ, on the other.

Undoubtedly, both *dessins d'enfants* and the modular group are amongst the most popular objects of modern mathematics; both are extensively covered in the literature. Thus, the modular group and its subgroups play a central rôle in the theory of modular forms, Moonshine theory, some aspects of number theory and hyperbolic geometry. *Dessins d'enfants*, apart from Grothendieck's original idea [83] connecting them (*via* Belyĭ maps) to the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, are related (*via* moduli spaces of curves and the Gromov–Witten theory) to topological field theories and integrable partial differential equations. Unfortunately, all of these fascinating topics are beyond the scope of this book. Our primary concern is a straightforward application of dessins to topology of trigonal curves and plane curves with deep singularities: the monodromy of such a curve can be computed, in a purely combinatorial way, in terms of its dessin; as a consequence, the monodromy group is a subgroup of Γ of genus zero and, using the well developed theory of such subgroups, we obtain numerous restrictions on the fundamental group of the curve and its more subtle geometric properties. This idea is summarized in Speculation 5.90 in chapter 5.

Principal results

Originally, my interest in *dessins d'enfants* was motivated by our work (joint with I. Itenberg and V. Kharlamov) on real trigonal curves and real elliptic surfaces and by my attempts to compute the fundamental groups of plane sextics. These two classes of objects remain the principal geometric applications of the theory. Here is a brief account of the most important statements found in the text.

Plane sextics We use skeletons to classify irreducible plane sextics with a triple singular point (including non-simple ones) and compute their fundamental groups, see Theorems 7.45, 8.1, and 8.2. Thanks to works by J.-G. Yang [166], I. Shimada [148], and the author [46], the classification of simple plane sextics is close to its completion. This classification relies on the global Torelli theorem for $K3$-surfaces and is not quite constructive; the 'visualization' of sextics, necessary for the detailed study of their geometry, remains an open problem. In the presence of a triple point, this problem is solved by means of the skeletons.

A brief survey of the known results concerning plane sextics and their fundamental groups is given in §7.2.1 and §7.2.3.

Another application in this direction is the classification up to deformation and the computation of the fundamental groups of singular plane quintics, see Theorems 7.49 and 7.50. This result is old, but its complete proof has never been published.

Universal trigonal curves and metabelian invariants The monodromy group of a non-isotrivial trigonal curve over a rational base is a subgroup of genus zero, and any subgroup $H \subset \Gamma$ of genus zero is realized by a certain *universal curve*, from which any other curve with the monodromy group subconjugate to H is induced, see §5.3.1 and Corollary 5.88. These facts impose strong restrictions on the fundamental group and relate the latter to some geometric properties of the curve. (I expect that there should be a reasonable description of all finite quotients of such groups.) As a first step, we obtain universal (independent of the singularities) bounds on the Alexander module of a trigonal curve, see Theorems 6.1 and 6.16. Then, as an illustration of Speculation 5.90(2), we establish a version of Oka's conjecture for trigonal curves, see Theorem 6.10, and classify the so-called *Z-splitting sections* of such curves, see Theorem 6.15: any Z-splitting section is induced from a certain universal one. This statement may have further implications to the study of tetragonal curves, hence plane sextics with $\mathbf{A}$ type singular points only.

Monodromy factorizations A long standing question, related to the study of the topology of algebraic and pseudo-holomorphic curves, is whether a simple factorization of a given monodromy at infinity is unique up to Hurwitz equivalence. We answer this question in the negative and show that the problem is much wilder than it might seem: in the group as simple as $\mathbb{B}_3$, the number of nonequivalent factorizations may grow exponentially in length, see Theorem 10.20. On the other hand, we give a complete classification of Γ-valued factorizations of length two, see Theorems 10.27, 10.30, and 10.32 (joint with N. Salepci). As a by-product, we show that any maximal real elliptic Lefschetz fibration is algebraic, see Theorem 10.88. With elliptic surfaces in mind, we also introduce a new invariant of monodromy factorizations, the so-called *transcendental lattice*, and study its properties, see §10.2.3.

As another application, we show that for extremal elliptic surfaces (see §10.3.1) and for a certain class of real trigonal curves, including M-curves (see Theorem 10.73), the topological and equisingular deformation classifications are equivalent. (Extremal elliptic surfaces are defined over algebraic number fields, and both classifications are also equivalent to the analytic classification in this case.)

Zariski k-plets We construct a few examples of exponentially large (with respect to appropriate discrete invariants) collections of nonequivalent objects sharing the same combinatorial data. The objects are: extremal elliptic surfaces (see Theorem 9.30), irreducible trigonal curves (the ramification loci of the surfaces above), real trigonal curves (see Example 10.85), and real Lefschetz fibrations (see Example 10.87). All of the examples are essentially based on Theorem 10.20; thus, in each case, the objects differ *topologically*, constituting the so-called *Zariski k-plets*. The trigonal curves also share such commonly used invariants as the fundamental group and transcendental lattice, see Addendum 9.36 and Theorem 9.31.

The transcendental lattice The j-invariant of an extremal elliptic surface is given by its skeleton. We show that the homological invariant, hence the surface itself, can be encoded by an orientation of the skeleton, see Theorem 9.1, and develop a simple algorithm computing the lattice of transcendental cycles and the Mordell–Weil group of the surface in terms of its oriented skeleton, see Theorem 9.6. More generally, the transcendental lattice and the torsion of the Mordell–Weil group of an arbitrary (not necessarily extremal) elliptic surface can be computed in terms of its homological invariant, regarded as a monodromy factorization, see Corollary 9.26. This algorithm leads us to the definition of transcendental lattice as an invariant of factorizations, see §10.2.3, and motivates a topological approach to the study of its arithmetic properties such as the discriminant form and parity.

Contents at a glance

The principal concepts are introduced in Part I: we discuss bipartite ribbon graphs and their relation to the subgroups of (appropriate quotients of) the free group $\mathbb{F}_2$, see chapter 1, the modular group Γ and closely related braid group $\mathbb{B}_3$, see chapter 2, and trigonal curves and elliptic surfaces, both complex and real, and their topological counterpart, the so-called *Lefschetz fibrations*, see chapter 3. For the reader's convenience, I have also included some background material that a topologist may not be familiar with and reproduced the proofs (or at least ideas of the proofs) of a few known statements, which are either difficult to find in the literature or closely related to the main subject. A separate section in chapter 1 deals with *pseudo-trees*, which are an important special class of skeletons used later on in the construction of various exponentially large examples and in the study of simple monodromy factorizations.

In chapter 4, we follow [60] and describe the (equivariant) equisingular deformation classes of (real) trigonal curves in terms of *dessins*—certain overdecorated embedded graphs which must be considered up to a number of moves and which can be rather difficult to handle. It turns out that, under some additional extremality assumptions, dessins can be replaced with skeletons, *i.q.* subgroups of the modular group, making their study feasible. In the real settings, this correspondence between skeletons and deformation classes of curves is made precise in §10.3.2.

In chapter 5, we recall the notion of *braid monodromy*, adjusted to the particular case of curves in ruled surfaces, and the Zariski–van Kampen theorem, computing the fundamental group of such a curve in terms of its monodromy group. The principal result here is a purely combinatorial computation of the braid monodromy of a trigonal curve in terms of its dessin/skeleton (see §5.2) and, as an upshot, a strong restriction on the monodromy group of a trigonal curve and the notion of *universal curve*. The two latter lead us to Speculation 5.90, which is copiously illustrated in chapter 6.

Part II deals with the geometric applications, both old and new. Here, the chapter names are self-explanatory. We compute and study the fundamental groups of trigonal curves and related plane curves (see chapters 6, 7, and 8), discuss the transcendental lattice of an extremal elliptic surface and work out a particular series of examples (see chapter 9), and make a few steps towards the understanding of Γ-valued monodromy factorizations and their applications to the topology of trigonal curves, elliptic surfaces, and Lefschetz fibrations (see chapter 10; for completeness, a few more or less classical results concerning the free groups $\mathbb{F}_n$, symmetric groups $\mathbb{S}_n$, and other braid groups $\mathbb{B}_n$ are also cited here).

Appendices collect the material that would not fit elsewhere. Appendix A contains a few assorted statements concerning integral lattices and quotient groups, especially the so-called *Zariski quotients*, which appear as the fundamental groups of algebraic curves. In Appendix B, for comparison and as a very simple application, we discuss bigonal (hyperelliptic) curves in Hirzebruch surfaces. Appendix C is a listing of the `GAP` code that handles technical details of some proofs. (Shorter *ad hoc* listings are included into the main text; all `GAP` files are available for download.) Appendix D is a glossary: we fix the notation and give a brief explanation of a few terms, with the selection based upon the author's own background and personal preferences.

Reading this book

Every effort has been made to produce a text as self-contained and cross-referenced as possible, so that it can be read starting at any point with only a very minimal background from the reader.

As usual, the end of a proof is marked with a □. Some statements are marked with a ◁, which means that either the statement is trivial (*e.g.*, most corollaries) or its proof has already been explained. If a statement is marked with a ▷, possibly followed by a list of references, its proof is omitted and the reader is directed to the literature. In most cases, the source is cited at the header.

We use the commonly accepted symbol $:=$ as a shortcut for 'is defined as'.

Most symbols typeset in a special font (bold, Gothic, calligraphic, *etc*.) represent objects or classes of objects introduced somewhere in the book; they should be found in §D.2. A brief explanation of other more or less common terms, notations, and concepts used throughout the whole text is given in §D.1.

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- the creators of `GAP` [76], a symbolic computation package without which some of the results of this book could not have been obtained in finite time,
- the creators of GLE, a software package that helps one replace frustrating mouse based picture drawing with the excitement of writing code and chasing bugs.

The final version of the manuscript was prepared during my sabbatical stay at *l'Instutut des Hautes Études Scientifiques*. I wish to extend my gratitude to this institution and its friendly staff for their hospitality and excellent working conditions.

I dedicate this book to my wife Ayşe Bulut: it is her constant support, patience, and understanding that make my work successful.

Ankara, December 2011

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