Syllabus for the Bilkent University Mathematics PhD Programme Qualifying Exam

Latest revision: 26 June 2006.

Written Exam: The duration of the exam is three hours. There are four subject areas. In each subject area, there are four questions. The candidate is to attempt two questions from each subject area. The syllabi for the four subject areas are as below.
Oral Exam: The candidate is to make a fifteen-minute presentation directed as if towards a general audience. This is to be followed by about fifteen minutes of questions. (The questions are likely to be requests for pedagogical clarification of basic concepts.) The assessment will be for teaching and communication. No credit will be awarded for specialist erudition.

ALGEBRA

Any overlap with lecture courses is accidental. The candidate is expected to master the material through independent study (using multiple sources). For theorems marked with a $\star$, the candidate is expected to be able to state and apply the theorem, but proof is not demanded.

Set theory: $\star$ Zorn's Lemma, $\star$ existence and uniqueness of the algebraic closure of a field.

Abstract group theory: Semidirect Products. Three Isomorphism Theorems, Holder–Jordan Theorem. Lagrange's Theorem. Sylow's Theorem. Burnside Basis Theorem for Finite p-Groups.

Galois Theory: $\star$ Eisenstein's Criterion, $\star$ Artin's Primitive Element Theorem, $\star$ Fundamental Theorem of Galois Theory, $\star$ Unsolvability of the Quintic. Calculation of Galois groups for polynomials such as $X^6 - 2$.

Ring Theory: Semisimple, Artinian and Noetherian modules. Semisimple, Artinian and Noetherian rings. Jacobson radical. $\star$ Artin–Wedderburn Structure Theorem.

Finite Group Representations: Frobenius Reciprocity. Maschke's Theorem. Orthogonality relations. Construction of ordinary character tables for groups such as S_4 and A_5.

Pervasive bibliography:
- M. Artin, "Algebra", (Prentise–Hall, New Jersey, 1991).
- I. M. Isaacs, "Algebra: a graduate course", (Brookes/Cole, California, 1994).

Concentrated bibliography:
- J. L. Alperin, R. B. Bell, "Groups and Representations", (Springer, Berlin, 1995).
- E. Artin "Galois Theory", revised by A. N. Milgram (Dover, New York, 1998).
- M. Aschbacher, "Finite group theory", 2nd ed. (C.U.P. 2000).
- G. James, M. Liebeck, "Representations and Characters of groups", 2nd ed. (C.U.P. 2001).
- T. Y. Lam, "A First Course in Non-Commutative Rings", (Springer, Berlin, 1991).
- W. R. Scott, "Group Theory", (Dover, New York, 1987).

Ambient bibliography:
- N. Bourbaki, "Algebra I, Chapters 1-3", (Springer, Berlin, 1989).
- B. L. van def Waerden, "A History of Algebra", (Springer, Berlin, 1985).

ANALYSIS

MATH 501 – REAL ANALYSIS I

1. Algebra of sets. Borel sets and Bair functions.
2. Measures and outer measures. Measurable and nonmeasurable sets.
3. Measurable functions. Lusin's and Egorov's theorems. Modes of convergence.
4. Integration. Limit theorems for Lebesgue integral.
5. Differentiation of Lebesgue integral. Functions of bounded variation. Absolutely continuous functions. Absolute continuity of indefinite Lebesgue integral.
6. Spaces of integrable functions.
7. Product measures. Signed measures. The Radon-Nikodym theorem. Hahn, Jordan and Lebesgue decompositions.

Books:

W.Rudin *Real and Complex Analysis*, McGraw-Hill Int.Editions, 1966
H.L.Royden *Real Analysis, 3rd Edition*, Macmillan Publ.Comp.,1988

MATH 502 – REAL ANALYSIS II

1. Metric and topological spaces. Completeness. Compactness. Compactness in concrete function spaces. Continuous mappings on compact metric spaces.
2. Hilbert spaces. Orthogonality. General Fourier series.
3. Banach spaces. Linear functionals and linear operators. Dual spaces. The Hahn-Banach theorem.
4. Category. Baire theorem. The Banach-Steinhaus theorem. The open mapping theorem.
5. Weak Topologies.

Books:

W.Rudin *Real and Complex Analysis*, McGraw-Hill Int.Editions, 1966
W.Rudin *Functional Analysis*, McGraw-Hill Int.Editions, 1973
E.Kreyszig *Introductory Functional Analysis with Applications*, John Wiley, 1989

COMPLEX ANALYSIS

A. Holomorphic Functions. Cauchy-Riemann equations, multi-valued functions, Cauchy theorem and integral formula, Morera theorem, power series representation, sequences of holomorphic functions, uniqueness theorem, open mapping theorem, maximum modulus principle, Cauchy theorem for multiply connected regions, Cauchy-type integrals.

B. Singularities. Classification, Laurent series, residue theorem, argument principle, Rouché theorem, residues at infinity, evaluation of integrals and sums.

C. Conformal Mapping. Preservation of angles, Schwarz-Pick lemma, mapping by Möbius transformations, normal families, Riemann mapping theorem, continuity at the boundary, simply connected regions.

D. Meromorphic and Entire Functions. Runge theorem on approximation by rational functions, Mittag-Leffler theorem, infinite products, Weierstrass factorization theorem, Jensen formula, Blaschke products, gamma and zeta functions.

E. Analytic Continuation. Regular and singular points, gap series, Schwarz reflection principle, continuation along curves, monodromy theorem.

F. Harmonic Functions. Maximum principle, mean value property, Harnack theorem, Poisson integral.

Books:

(Chapters 10–16 of [**1**] covers most of the topics, but the sections 11.15–11.32, 12.11–12.14, 14.10–14.15, 15.25–15.27, 16.17–16.22 are less relevant. The other references also contain sections more or less equivalent to these. For a few topics not covered by the above (such as multi-valued functions, residues at infinity, gamma and zeta functions), one can consult the other references.)

[**1**] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.

[**2**] R. E. Greene & S. G. Krantz, *Function Theory of One Complex Variable*, 2nd ed., American Mathematical Society, 2002.

[**3**] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer, 1978.

[**4**] T. W. Gamelin, *Complex Analysis*, Springer, 2001.

APPLIED MATHEMATICS

MATH 543 Methods of Applied Mathematics I

1. Function spaces, completeness, square integrable functions. Orthogonal sets of vectors and the Bessel inequality. Basis and Parseval's relation. Weierstrauss' theorem. Classification of orthogonal polynomials, classical orthogonal polynomials. Trigonometric series, generalized functions

2. Second order differential equations. Fundamental solutions. method of variation of constants. Generalized Green's identity: Adjoint operators and adjoint boundery conditions. The method of Green's function. The Sturm–Liouville problem, classification of singular points. The Frobenius method: The series solutions of linear DEs. Fuchsian differential equations, the hypergeometric function. Solutions of DEs by integral representations. Integral representations of hypergeometric functions.

3. Regular perturbations, Poincare–Lindstedt method. Singular purturbations, boundery layer problems. WKB approxinmation, asymptotic expansion of integrals.

4. Calculus of variations, necessary condition, Euler–Lagrange equations, Lagrange functions depending on higher derivatives, null Lagrange functions, lagrange function of given DE. Lagrange function with several dependent variables. Iso-perimetric problems.
Books: Dennery and Krzywicki for 1 and 2. Logan for 3. Logan and Hildebrand for 4.
For more details see the course web site
http:// www.fen.bilkent.edu.tr/gurses/applied1.html.
For Qualifying Exam questions, see those *assigned exercises* at this web site that are related to the above subjects.

MATH 544 Methods of Applied Mathematics II

1. Partial differential equation models (G–L, Copson, Logan). Classification of 2nd order partial differential equations (G–L, Copson) The Cauchy–Kowalewsky theorem (Copson). Linear, quasi-linear, half-linear equations (Copson). Initial and BV problems for the wave equation (G–L). Some existence and uniquenessn theorems (G–L). First order hyperbolic systems. The Riemann Method (G–L, Copson). Exact solutions, uniqueness, maximum-minimum theorems of hyperbolic type of (the wave) equation, of paranolic type of (the heat) equation, of elliptic type of (the Laplace) equation (G–L, D–K).

2. Integral equations and Green's functions (G–L, Hildebrand). Sturm–Liouville problems, Neumann series, Hilbert–Schmidt theory of singular integral equations.

3. Stability and bifurcation (Logan, Boyce–DiPrima Ch. 9). One-dimensional problems, two-dimensional problems. Hydrodynamics stability.

DK = Dennery and Krzywicki, G–L = Roland B Guenter and J W Lee.
For more details see the web site
http:// www.fen.bilkent.edu.tr/gurses/applied2.html.
For Qualifying Exam questions, see those *assigned exercises* at this web site that are related to the above subjects.

GEOMETRY AND TOPOLOGY

A. Geometry

1. Differentiable Manifolds, Differentiable Functions and Mappings: Differentiable manifolds, differentiable functions and mappings, rank of a mapping, immersions, submersions, submanifolds and embeddings.

2. Vector Fields on Manifolds: Tangent space at a point of a manifold, the differential of a differentiable mapping, vector fields, Lie bracket of vector fields.

3. Tensors and Tensor Fields on Manifolds: Tensors, tensor fields and differential forms, pull-back of a differentiable mapping by differentable mapping, exterior differentiation, Riemannian metric on manifolds, orientation on manifolds, volume element.

4. Integration on Manifolds: Integration on manifolds, manifolds with boundary, boundary orientation of the boundary of a manifold, Stokes's Theorem.

References:
Boothby, W., *An Introduction to Differentiable manifolds and Riemannian Geometry* (sections: III.1-III.5, IV.1-IV.2, V.1-V.8, VI.1, VI.2, VI.4, VI.5)
Warner, F., *Foundations of Differentiable Manifolds and Lie Groups*, (sections: 1.1-1.42, 2.1-2.23, 4.1- 4.9)

B. Topology

1. Topological spaces and continuous functions: Topological spaces, basis and subbasis, subspace topology, continuous functions, product topology, metric topology, quotient topology.

2. Compactness: Compact spaces, compact sets in Rn, Heine–Borel Theorem, Tychonoff Theorem, limit-point compactness, sequential compactness, compactness in metric spaces, local compactness and one-point compactification.

3. Connectedness: Connected spaces, path-connected spaces, components, local connectedness, local path-connectedness.

4. Separation and Countability Properties: T0, Hausdorff, regular, normal spaces; Uryshon Lemma, Tietze Extension Theorem, countability properties; Lindelf, separable, countably compact spaces

References:
Munkres J., *Topology, a First Course*, (sections: 2.1-2.10, 3.1-3.8, 4.1- 4.3)
Willard S., *General Topology*, (sections: 2, 3, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19. In section 19 only one-point compactification is included.)

C. Algebraic Topology:

Fundamental group, Van Kampen's Theorem, covering spaces. Singular Homology: Homotopy invariance, homology long exact sequence, Mayer- Vietoris sequence, excision. Cellular homology. Homology with coefficients. Simplicial homology and the equivalence of simplicial and singular homology. Axioms of homology. Homology and fundamental group. Simplicial approximation. Cohomology groups, Universal Coefficient Theorem, cohomology of spaces.

Products in cohomology, Knneth formula. Poincar Duality. Universal Coefficient Theorem for homology. Homotopy groups.

Main Reference: A. Hatcher, *Algebraic Topology* (2000).
Other References: J. Munkres, *A First Course in Topology* (Chapter 8) and *Elements of Algebraic Topology.* G. Bredon, *Geometry and Topology.* J. J. Rotman, *An Introduction to Algebraic Topology.* E. Spanier, *Algebraic Topology.* M. Greenberg and J. Harper. *Algebraic Topology.* W. S. Massey, *A Basic Course in Algebraic Topology.*

D. Algebraic Geometry (Math 551):

Theory of algebraic varieties: Affine and projective varieties, dimension, singular points, divisors, differentials, Bezout's theorem.

Main Reference: R. Hartshorne, *Algebraic Geometry*, (Chapter 1).
Other References: I. R. Shafarevitch, *Basic Algebraic Geometry* (Part 1), K. Smith–L. Kahanp *et al.*, *An Invitation to Algebraic Geormetry*, K. Ueno, *An Introduction to Algebraic Geometry*, P. Griffiths, J.Harris, *Principles of Algebraic Geometry*, (Chapter 0).

E. Differential Geometry I (Math 545):

Lie derivative of tensor fields. Connections, covariant differentiation of tensor fields, parallel translation, holonomy, curvature, torsion.

Levi-Civita (or Riemannian) connection, geodesics, normal coordinates. Sectional curvature, Ricci curvature and scalar curvature, Schur's theorem. Jacobi Fields, conjugate points. Isometric immersions, the second fundamental form, formulae of Gauss and Weingarten. Equations of Gauss, Codazzi and Ricci. Metric and geodesic completeness, the Hopf-Rinow theorem. Variations of the energy functional.

Main Reference: Manfredo P. Do Carmo, *Riemannian Geometry* 1993 (Chapters 1-7, 9).
Other References: William M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry* 1986, S. Kobayashi– K. Nomizu, *Foundations of Differential Geometry vols. I, II* by S.Kobayashi-K.Nomizu (1963), T. Aubin, *Course in Differential Geometry* 2000, T. Sakai, *Riemannian Geometry* (1996).