Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

May 2005

Problem: Find the maximum and minimum of $|a-b|+|b-c|+|c-a|$, if integer numbers a, b, and c satisfy the following relation:

$$(a-b)^3 + (b-c)^3 + (c-a)^3 = 60$$

Solution: Note that

$$(a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a).$$

Therefore, $(a-b)(b-c)(c-a) = 20$. Put $a-b = x, b-c = y, c-a = z$. Now we have a system of two equations

$$\begin{cases} xyz = 20 \\ x+y+z = 0. \end{cases}$$

The integer solutions are $(5, -4, -1), (-5, 4, 1)$ and their permutations. Thus, the expression $|a-b|+|b-c|+|c-a|$ is a constant, and the maximum and minimum of this expression is 10.