Bilkent University
Department of Mathematics

Problem Of The Month

December 2023

Problem:

There are several red and several white boxes on the table, each of these boxes contains at least one ball. A positive integer number not exceeding 1111 is written on each of these balls.

† Any two boxes contain different number of balls.

†† No box contains two balls with the same number.

††† For each $1 \leq i \leq 1111$ there is at most one red box containing ball with number i.

†††† For each $1 \leq i \leq 1111$ there is at most one white box containing ball with number i.

Find the maximal possible number of boxes on the table.

Solution: Answer: 66.

Suppose that the total number of boxes is n and these boxes contain $a_1 < a_2 < \cdots < a_n$ balls. Then

$$2 \cdot 1111 \geq \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

The largest integer n satisfying the above inequality is $n = 66$. Now one can construct a proper example for $n = 66$ when there are 46 red boxes containing $1, 2, \ldots, 15, 16, 18, 19, \ldots, 46, 47$ balls and 20 white boxes containing $17, 48, 49, \ldots, 46, 47$ balls, respectively since here the total number of balls in red boxes is 1111 the and the total number of balls in white boxes is 1100.