Bilkent University
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Problem Of The Month

November 2023

Problem:

Let P be a polynomial of degree n with real coefficients such that at least n of its coefficients coincide and at most one of its coefficients is zero. Given that all n roots of P are real numbers find the maximal possible value of n.

Solution: Answer: The maximal possible value is $n = 4$.

The polynomial

$$P(x) = x^4 + x^3 - 4x^2 + x + 1 = (x-1)(x-1)(x + \frac{3+\sqrt{5}}{2})(x + \frac{3-\sqrt{5}}{2})$$

satisfies problem conditions. Let us show that there is no polynomial of degree $n \geq 5$ satisfying the conditions. Without loss of generality we assume that at least n coefficients of the polinomial are 1.

Let $x_1, x_2, \ldots, x_n$ be the roots and S_k be the sum of k-wise sums of the roots:

$$S_1 = x_1 + \cdots x_n, S_2 = x_1x_2 + \cdots + x_{n-1}x_n, \ldots, S_n = x_1x_2 \cdots x_n$$

By Vieta theorem we have

$$S_1^2 - 2S_2 = \sum_{i=1}^{n} x_i^2 \geq 0.$$

If the first three coefficients of P are 1 then we get a contradiction with the last inequality:

$$S_1 = -1,\ S_2 = 1 \text{ and } S_1^2 - 2S_2 = -1 < 0.$$

Therefore, one of the first three coefficients is not 1. Then the free coefficient is 1 and 0 is not a root of P. In this case we have

$$\left(\frac{S_{n-1}}{S_n}\right)^2 - 2\left(\frac{S_{n-2}}{S_n}\right) = \sum_{i=1}^{n} \frac{1}{x_i^2} > 0.$$

If the last three coefficients of P are 1 then we get a contradiction with the last inequality:

$$S_{n-2} = -S_{n-1} = S_n \text{ ve } \left(\frac{S_{n-1}}{S_n}\right)^2 - 2\left(\frac{S_{n-2}}{S_n}\right) < 0.$$

Since at least n coefficients of P out of $n+1$ are 1 when $n \geq 5$ either first three or last three coefficients of P should be 1, a contradiction.