Bilkent University
Department of Mathematics

Problem Of The Month

June 2023

Problem:

Find all pairs (p, q) satisfying

$$p(p^4 + p^2 + 10q) = q(q^2 + 3),$$

where p and q are prime numbers.

Solution: Answer: $(p, q) = (2, 5)$.

If $p = q$ then left hand side is greater than right hand side, so we may assume that $p \neq q$. If $p > 3$, then $p \mid q^2 + 3$ and hence -3 is a quadratic residue in $\pmod{p}$ and consequently $p \equiv 1 \pmod{3}$. Then, looking at modulo 3 we get

$$p(p^4 + p^2 + 10q) \equiv q + 2 \pmod{3}$$

while $q(q^2 + 3) \equiv q \pmod{3}$, by Fermat's little theorem or just by direct check, a contradiction.

Checking the values $p = 2, 3$ we find out that the only solution is $(p, q) = (2, 5)$.