Bilkent University
Department of Mathematics

Problem Of The Month

December 2022

Problem:

A positive integer is said to be square free if it is not divisible by a square greater than 1. We say that a positive integer is *simple* if each of the numbers $a, a+1$ and $a+2$ is square free. Are there infinitely many simple integers?

Solution: Answer: There are infinitely many simple integers.

Let n be a fixed positive integer. Let us consider the set $A(n)$ of all integers from $[1, 4n^2]$ not divisible by 4. For each integer $k \in A(n)$ divisible by 9 we have $k = 9, 18, 27 \pmod{36}$. The total number of such integers n is not greater than $3 \cdot \dfrac{4n^2}{36} + 3$. For each integer $k \in A(n)$ divisible by 25 we have $k = 25, 50, 75 \pmod{100}$. The total number of such integers n is not greater than $3 \cdot \dfrac{4n^2}{100} + 3$. By continuing similar argument till integer $k \in A(n)$ divisible by $(2n-1)^2$ we get that the total number of not square free integers in $A(n)$ is not greater than

$$3n^2\left(\frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n-1^2}\right) + 3(n-1) = \frac{3n^2}{2}\left(\frac{2}{9-1} + \frac{2}{25-1} + \cdots + \frac{2}{(2n-1)^2-1}\right) + 3(n-1)$$

$$= \frac{3n^2}{2}\left(\frac{1}{3-1} - \frac{1}{3+1} + \frac{1}{5-1} - \frac{1}{5+1} \cdots + \frac{1}{(2n-1)-1} - \frac{1}{(2n-1)+1}\right) + 3n < \frac{3n^2}{4} + 3n$$

There are n^2 disjoint triples of consecutive integers in $A(n)$: $(1,2,3)$, $(5,6,7)$, $(9,10,11)$, $\ldots$ By the inequality above at least $n^2 - (\dfrac{3n^2}{4} + 3n) = \dfrac{n^2}{4} - 3n$ of these triples are constituted only by square free integers. Since $\dfrac{n^2}{4} - 3n$ is unboundedly increasing while $n \to \infty$ we are done.