Bilkent University
Department of Mathematics

Problem Of The Month

October 2022

Problem:

Each of n guests at a party wears a hat of one of three colours. A guest is said to be *lucky* if at least two of its friends in the party wear differently coloured hats. Find all values of n for which it is **always** possible to choose a guest and to change its hat to hat of one of the remaining two colours such that the total number of lucky guests will not be decreased.

Solution: Answer: n is not a multiple of 3.

Consider a graph G where vertices represent guests and two vertices are adjacent if and only if guests corresponding to these vertices are friends. We will colour vertices to colours a, b, c according to hat colours of corresponding guests. Let us show that if n is any multiple of 3 then there are examples when any colour change leads to lucky guest decrease. If G is a union of disjoint cycles of length 3 and vertices of each cycle are coloured differently like a, b, c then any recolouring will lead to lucky guest decrease. Note that this is not the only possible example: the colouring a, a, b, b, c, c of a cycle of length 6 also satisfies conditions.

Now let us show that if n is not a multiple of 3 then it is always possible to find a guest satisfying the conditions. Each lucky guest has at least two friends. Note that a lucky guest having exactly 2 friends can be made unlucky in two different ways: the hat colours of her friends should be changed from (a, b) to either (a, a) or (b, b). A lucky guest having at least 3 friends can be made unlucky in just one way: the hat colour of her uniquely determined friend should be changed to uniquely determined colour. There are $2 \cdot n$ possibilities to change hat colour of one guest. If each of these changes produces some unlucky guest then each person should have exactly two friends. Then G will be decomposed into disjoint cycles. In each cycle each vertex should have differently coloured neighbours and any recolouring of any vertex produces a vertex with same coloured neighbours. Therefore, the colouring of each cycle of G should be as $a, \cdot, b, \cdot, c, \cdots\cdots, a, \cdot, b, \cdot, c, \cdot$ and consequently the length of each cycle is a multiple of 3. Since n is not a multiple of 3 it is impossible. Done.