Bilkent University
Department of Mathematics

Problem Of The Month

February 2022

Problem:

During of the school year an instructor asked 2022 problems in the class and the number of students who could not solve any particular problem was at most two. The instructor wants to divide the 2022 problems into three folders of 674 problems and give each folder to a student who solved all 674 problems in that folder. Find the minimum number of students in the class that makes it possible in all possible situations.

Solution: Answer: $n = 5$.

Assume there are four students S_i, $i = 1, 2, 3, 4$: S_1 and S_2 solved the same half of the problems and S_3 and S_4 solved the other half. Then readily the teacher can not divide 2022 problems into three folders and give each folder to a student who solved all 674 problems in the folder.

Let us show that the required partition is always possible for 5 students. Assume that there are five students S_1, S_2, S_3, S_4, S_5. For $1 \leq i \leq 5$, let n_i be the number of problems that are not solved only by S_i and for $1 \leq i < j \leq 5$, $n_{ij} = n_{ji}$ be the number of problems that are not solved neither by S_i nor by S_j. Since there are at most two students who did not solve any given problem, we have

$$\sum_i n_i + \sum_{i<j} n_{ij} \leq 2022.$$

For $1 \leq i \leq 5$, let w_i be the number of problems not solved by S_i: $w_i = n_i + \sum_{i \neq j} n_{ij}$.

Claim. There are $1 \leq p < q < r \leq 5$, such that $w_i \leq 1348$ and $n_{ij} \leq 674$ for $i, j \in \{p, q, r\}$ and $i \neq j$.

Proof. Assume that there are three students, say S_1, S_2, S_3 who solved less than 674 problems each. Then $w_i \geq 1349$ for $i = 1, 2, 3$ and

$$3 \cdot 1349 \leq w_1 + w_2 + w_3 \leq 2(\sum_i n_i + \sum_{i<j} n_{ij}) \leq 2 \cdot 2022$$

gives a contradiction. Therefore, there are at least three students with $w_i \leq 1348$. Without loss of generality, we assume that these include S_1, S_2, S_3.

If each n_{12}, n_{13}, n_{23} is less than or equal 674, then the Claim is proven. If not, then since $w_i \leq 1348$ for $i = 1, 2, 3$, at most one of n_{12}, n_{13}, n_{23} can be greater than 674. Let us assume that $n_{12} > 674$. Then $n_{13} < 674, n_{23} < 674$ and also $w_4 < 1348, w_5 < 1348$. Applying the same reasoning to the triple w_2, w_3, w_4 we conclude that $n_{34} > 674$. Now we have $w_i \leq 1348$ for $i = 1, 3, 5$, and $n_{13} < 674, n_{15} < 674, n_{35} < 674$. The Claim is proven.

Let S_p, S_q, S_r be the three students for which the Claim is true. We will assign the problems one by one to one of the students S_p, S_q, S_r who solved it. Suppose we are about to assign problem P. Since we have three students, P is solved by at least one of them. If there is a student who solved P and has less than 674 problems assigned to her up to this point, we assign P to her. If this is not the case, then we proceed as follows.

If S_p is the only student who solved P, but she has already 674 problems assigned to her, and both S_q and S_r have less than 674 problems assigned to them, then we will reassign one of the problems P' from S_p to S_q or S_r, and assign P to S_p. If it is not possible for any P' then $n_{q,r} > 674$, a contradiction.

If S_p solved P, but she and S_q both have already 674 problems assigned to them, then as above we can reassign a problem P' from S_p to S_q or S_r, and assign P to S_p. If in this step we can not reassign any problem to P_r then we will try to reassign a problem P' from S_p to S_q and also reassign a problem P'' from from S_q to S_r (and of course, assign P to S_p). If it is not possible for any P'' either, then $w_z \geq 1349$, a contradiction.