Bilkent University
Department of Mathematics

Problem Of The Month

January 2022

Problem:

A pair (n, k), where $n > k \geq 0$ are integers is said to be *strange* if for any $a \equiv k \pmod{n}$ and any positive integer m the expression

$$\frac{a^m + 3^m}{a^2 - 3a + 1}$$

is not integer. Find the smallest n such that (n, k) is strange for some k.

Solution: Answer: $n = 11$.

Let $a \equiv 5 \pmod{11}$. Then $11 | a^2 - 3a + 1$. On the other hand since

$$\{5^m, m = 1, 2, \dots\} = 5, 3, 4, 9, 1, 5, \dots$$

$$\{3^m, m = 1, 2, \dots\} = 3, 9, 5, 4, 1, 3, \dots$$

$a^m + 3^m = 5^m + 3^m$ is never integer. Therefore, $(11, 5)$ is strange.

Let $f(a, m) = \dfrac{a^m + 3^m}{a^2 - 3a + 1}$. Now we prove that (n, k) in not strange for $1 \leq n \leq 10$ by showing that for each $1 \leq n \leq 10$ and each $0 \leq k < n$ there exist $a \equiv k \pmod{n}$ such that $f(a, m)$ is integer.

For $a = 0, 1, 2, 3$ the expression $a^2 - 3a + 1 = \pm 1$ and hence $f(a, m)$ is always integer. Therefore, (n, k) is not strange for any n and $k = 0, 1, 2, 3$.

Note that $f(4, 2), f(6, 9), f(7, 2), f(8, 20)$ are integers. Therefore, (n, k) is not strange for any n and $k = 4, 6, 7, 8$.

Since $f(-1, 2)$ is integer $(6, 5)$ is not strange.
Since $f(12, 9)$ is integer $(7, 5)$ is not strange.
Since $f(-3, 1)$ is integer $(8, 5)$ is not strange.
Since $f(-4, 14)$ is integer $(9, 5)$ is not strange.
Since $f(-5, 20)$ is integer $(10, 5)$ is not strange.
Since $f(-1, 2)$ is integer $(10, 9)$ is not strange.

We are done.