**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

December 2021

**Problem:**

Find all primes $p$ for which there exist an odd integer $n$ and a polynomial $Q(x)$ with integer coefficients such that the polynomial

$$1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i)$$

has at least one integer root.

**Solution:** Answer: $p = 2$.

Let $P(x) = 1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i)$. For $p = 2$, $n = 1$ and $Q(x) = 2x + 1$ are suitable, since the corresponding polynomial has a root -1:

$$1 + 2 \cdot 1^2 + (2 \cdot (-1) + 1)(2 \cdot (-1^2) + 1) = 0.$$

Let us show that for all primes $p \geq 3$ no suitable $n$ and $Q(x)$ exist. By Fermat's little theorem $x^i = x^{i+p-1}$ and hence $Q(x^i) = Q(x^{i+p-1})$ for all $1 \leq i \leq p-1$. Therefore, $P(x) = 0$ in modulo $p$ leads to

$$0 \equiv 1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i) \equiv 1 + (\prod_{i=1}^{p-1} Q(x^i))^2$$

Thus -1 is a quadratic residue modulo $p$ and hence $p \equiv 1 \pmod 4$. Then $P(x) = 0$ in modulo 4 leads to

$$0 \equiv 1 + 1 + \prod_{i=1}^{2p-2} Q(x^i) \equiv 2 + (\prod_{i=1}^{p-1} Q(x^i))^2 \qquad (1)$$

Note that for integer $x$ and positive $i, j$ the values $Q(x^i)$ and $Q(x^j)$ have the same parity. Therefore, both cases when all $Q(x^i)$ are odd and all $Q(x^i)$ are even we get a contradiction with (1). Done.