Bilkent University
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Problem Of The Month

November 2021

Problem:

Let $1 < a_1 < a_2 < \cdots < a_n$ and $b_1, b_2, \ldots, b_n$ be integers such that for any integer M, at least one of the numbers

$$\frac{M - b_i}{a_i},\ i = 1, 2, \ldots, n$$

is an integer. Find the smallest possible value of n.

Solution: Answer: 5.

Every integer N satisfies at least one of the congruences $N \equiv 0 \pmod{2}$, $N \equiv 1 \pmod{3}$, $N \equiv 3 \pmod{4}$, $N \equiv 5 \pmod{6}$, $N \equiv 9 \pmod{12}$. Therefore n can be 5. Let us show that $n \leq 4$ is not possible.

Let $1 < k_1 \leq k_2 \leq \cdots \leq k_n$ and $a_1, a_2, \ldots, a_n$ be integers, and let $K = \operatorname{lcm}(k_1, k_2, \ldots, k_n)$. Since at most $\dfrac{K}{k_1} + \dfrac{K}{k_2} + \cdots + \dfrac{K}{k_n}$ integers from 1 to K can satisfy at least one of the congruences $x \equiv a_i \pmod{k_i}$ for $1 \leq i \leq n$, we must have $\dfrac{1}{k_1} + \dfrac{1}{k_2} + \cdots + \dfrac{1}{k_n} \geq 1$ if every integer satisfies at least one of these congruences.

Now assume that $1 < k_1 < k_2 < \cdots < k_n$ and $a_1, a_2, \ldots, a_n$ satisfy the condition of the problem and that $n \leq 4$ has the smallest possible value. If $k_1 = 3$, then

$$\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20} < 1\,.$$

Therefore $k_1 = 2$. Without loss of generality we may assume that $a_1 = 1$. For $2 \leq i \leq n$, let $k'_i = k_i$ and $a'_i \equiv 2^{-1} a_i \pmod{k_i}$ if k_i is odd, and let $k'_i = \dfrac{k_i}{2}$ and $a'_i = \dfrac{a_i}{2}$ if k_i is even. The integers $k'_2, \ldots, k'_n$ and $a'_2, \ldots, a'_n$ satisfy the condition of the problem except that k_i' might not be distinct. Therefore by the minimality of n, we must have $n = 4$ and $\{k_2, k_3, k_4\} = \{2m+1, 4m+2, k\}$.

If k is odd, then $\{k_2', k_3', k_4'\} = \{2m+1, 2m+1, k\}$ and $\dfrac{2}{2m+1} + \dfrac{1}{k} \geq 1$. Since

$$\frac{2}{2m+1} + \frac{1}{k} \leq \frac{2}{3} + \frac{1}{5} = \frac{13}{15} < 1\,,$$

this is not possible.

If k is even, then $\{k_2', k_3', k_4'\} = \{2m+1, 2m+1, \dfrac{k}{2}\}$ and $\dfrac{2}{2m+1} + \dfrac{2}{k} \geq 1$. If $2m+1 \geq 5$ or $2m+1 = 3$ and $k \geq 8$, we get contradictions because

$$\frac{2}{5} + \frac{2}{4} = \frac{9}{10} < 1 \quad \text{and} \quad \frac{2}{3} + \frac{2}{8} = \frac{11}{12} < 1\,.$$

The only remaining case is $2m+1 = 3$ and $k = 4$. This gives $\{k_2', k_3', k_4'\} = \{3, 3, 2\}$. Since the integers in a congruence class modulo 3 cannot be all even or all odd, this also leads to a contradiction. We are done.