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Problem Of The Month

September 2021

Problem:

There are 777 points located on a circle ω each coloured into one of the colours $1, 2, \ldots, k$. For each of these points and for each colour $1 \leq r \leq k$ there exists an arc of ω containing this point such that at least half of the points located on this arc are r coloured. Find the maximal possible value of k.

Solution: Answer: 3.

The colouring of each point into one of three colours such that any three consecutive points ate differently coloured satisfies conditions.

Let us show that $k \leq 3$. Each arc will be represented by its extreme points: the arc containing points $A, B, \ldots, Z$ in clockwise order will be denoted by (A, Z). For a colour r and a point A not coloured r let $l_A(r) = (B, C)$ be the arc satisfying the condition with the smallest number of points on it.

Claim 1. If $l_A(r) = (B, C)$ then either $(B, C) = (B, A)$ or $(B, C) = (A, C)$.

Assume the contrary. Let the arc (B, A) contains m points and b of them are coloured r. Let the arc (A, C) contains n points and c of them are coloured r. Then $\frac{b}{m} < \frac{1}{2}$, $\frac{c}{n} < \frac{1}{2}$ and we readily get $2b \leq m - 1$, $2c \leq n - 1$. Therefore,

$$2b + 2c \leq m + n - 2 < m + n - 1 \text{ and consequently } \frac{b+c}{m+n-1} < \frac{1}{2},$$

which contradicts with the definition of $l_A(r) = (B, C)$.

Now without loss of generality suppose that for a colour r and a point A not coloured r $l_A(r) = (A, C)$. By the definitions, C is r coloured.

Claim 2. Let the arc (A, C) contains n points and c of them are coloured r. Then $\frac{c}{n} = \frac{1}{2}$.

Assume the contrary: $\frac{c}{n} > \frac{1}{2}$. Then $2c > n$ and $2c \geq n + 1$ and herewith

$$2c - 2 \geq n - 1 \quad \text{ve} \quad \frac{c-1}{n-1} \geq \frac{1}{2}.$$

which contradicts with the minimality of (A, C).

Claim 3. For a colour r and points A and B ($A \neq B$) not coloured r let $l_A(r) = (A, C)$ and $l_B(r) = (B, D)$. Then $C \neq D$.

Assume the contrary: $C = D$. Without loss of generality suppose that $B \in (A, C)$. When we move from A in clockwise direction let T be the last point before B. By Claim 2, the ratios of points coloured r in both arc AC and arc BC are equal to $\frac{1}{2}$. Therefore, the ratio of of points coloured r in (A, T) also is equal to $\frac{1}{2}$, which contradicts with the minimality of (A, C).

Let $k > 3$. Then there is a colour r such that the number of points coloured r is at most $\lfloor 777/4 \rfloor = 194$. For each point A not coloured r by Claim 1 $l_A(r)$ is either (F, A) or (A, E), where points E and F are coloured r. By Claim 3, any point E coloured r can be the endpoint of at most two such arcs. Since there are at least $583 > 2 \cdot 194$ points not coloured 1 we get a contradiction. Done.