Bilkent University
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Problem Of The Month

November 2020

Problem:

Suppose that positive real numbers $a_{i,j}$, $i, j \in \{1, 2, \dots, 2020\}$ for each pair (i, j) satisfy $a_{i,j}a_{j,i} = 1$. For each $i = 1, \dots, 2020$ let $c_i = \sum_{k=1}^{2020} a_{k,i}$. Find the maximal possible value of $\sum_{i=1}^{2020} \frac{1}{c_i}$.

Solution: Answer: 1.

Let $c = \sum_{j=1}^{n} \frac{1}{c_j}$. If $a_{i,j} = 1$ for all (i, j) then $c = 1$. Let us show that $c \leq 1$. By Cauchy-Schwarz inequality we have

$$\sum_{j=1}^{n} \frac{x_j^2}{a_{ji}} \geq \frac{\left(\sum_{j=1}^{n} x_j\right)^2}{\sum_{j=1}^{n} a_{ji}} \tag{1}$$

for every i and positive real numbers $x_1, \dots, x_n$. Since $a_{ij}a_{ji} = 1$ for every i and j, letting $x_j = \frac{1}{c_j}$ in (1) yields

$$\sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^2 \frac{1}{c_i} \tag{2}$$

for every i. By adding up the inequality in (2) for every i we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^2 \sum_{i=1}^{n} \frac{1}{c_i} = c^3. \tag{3}$$

On the other hand, as

$$(4) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{ij}}{c_j^2}=\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{a_{ij}}{c_j^2}=\sum_{j=1}^{n}\left(\frac{1}{c_j^2}\sum_{i=1}^{n}a_{ij}\right)=\sum_{j=1}^{n}\left(\frac{1}{c_j^2}c_j\right)=\sum_{j=1}^{n}\frac{1}{c_j}=c,$$

inequality in (3) and equation (4) imply $c \geq c^3$. Then, as c is positive, we see that $c \leq 1$.

Solution 2. We will prove the inequality by induction over n. For $n = 2$, let $a_{1,2} = a$, then $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{1+a}+\frac{1}{1+1/a}=1$. So the inequality holds with equality.

Suppose that the inequality holds for $n = k$: $\sum_{i=1}^{k}\frac{1}{c_i}\leq 1$. We will prove it for $n = k + 1$. Note that by Cauchy-Schwarz inequality, for any $c, a, x \in \mathbb{R}$ we have $(c+a)(\frac{x^2}{c}+\frac{1}{a}) \geq (x+1)^2$ or

$$\frac{1}{c+a}\leq\left(\frac{x^2}{c}+\frac{1}{a}\right)(x+1)^{-2}$$

Therefore, for any x we get

$$\sum_{i=1}^{k}\frac{1}{c_i+a_{k+1,i}}\leq\sum_{i=1}^{k}\left(\frac{x^2}{c_i}+\frac{1}{a_{k+1,i}}\right)(x+1)^{-2}\leq\frac{x^2+\sum_{i=1}^{k}a_{i,k+1}}{(x+1)^2}$$

Now by choosing $x=\sum_{i=1}^{k}a_{i,k+1}$ we get

$$\sum_{i=1}^{k+1}\frac{1}{\sum_{j=1}^{k+1}a_{j,i}}=\sum_{i=1}^{k}\frac{1}{c_i+a_{k+1,i}}+\frac{1}{\sum_{j=1}^{k}a_{j,k+1}+a_{k+1,k+1}}\leq\frac{x^2+x}{(x+1)^2}+\frac{1}{x+1}=1.$$