Bilkent University
Department of Mathematics

Problem Of The Month

October 2020

Problem:

In a country consisting of 1001 cities there are two way flights between some n pairs of cities. It is observed that for any two cities A and B there is a sequence of 1000 flights starting at A ending at B and visiting each of the remaining cities exactly once. Find the minimal possible value of n.

Solution: Answer: 1502.

We reformulate the problem in terms of the graph theory. Find the minimal possible number of edges in a graph on 1001 vertices if there is a Hamiltonian path between any pair of vertices. Let us show that the degree of any vertex is greater than 3. Suppose the degree of some vertex A is 2 and A is adjacent to vertices B and C. Then the is no Hamiltonian path starting at B and ending at C. Similarly the degree of A can not be 0 and 1. Thus, the sum of all degrees is at least 3003 and the total number of edges is at least $\lceil \frac{3003}{2} \rceil = 1502$.

Now we construct a graph G with 1502 edges satisfying the conditions. Let us place the vertices of G to the points with the coordinates

$$(1,0),(2,0),(3,0),\ldots,(500,0),(1,1),(2,1),(3,1),\ldots,(500,1) \text{ and } (0,0).$$

Suppose that for each $i = 1,2,\ldots,499$ there is an edge between $(i,0)$ and $(i+1,0)$, between $(i,1)$ and $(i+1,1)$. Also suppose that for each $i = 1,2,\ldots,500$ there is an edge between $(i,0)$ and $(i,1)$. Finally, suppose that there are 4 edges connection the vertex $(0,0)$ with vertices $((0,1),(1,0),(500,0)$ and $(500,1)$. Let us show that the graph G with 1502 edges satisfies the conditions of the problem.

The Hamiltonian path between vertices $(2m,0)$ and $(2n+1,1)$, $1 < m < n < 250$ is

$$(2m,0) \to (2m+1,0) \to \cdots \to (2n,0) \to (2n,1) \to (2n-1,1) \to \cdots$$

$$\to (2m,1) \to (2m-1,1) \to (2m-1,0) \to (2m-2,0) \to (2m-2,1) \to zigzag$$

$$\to (2,0) \to (2,1) \to (1,1) \to (1,0) \to (0,0) \to (500,1) \to (500,0) \to (499,0)$$

$$\to (499,1) \to zigzag \to (2n+2,1) \to (2n+2,0) \to (2n+1,0) \to (2n+1,1).$$

The Hamiltonian path between vertices $(0,0)$ and $(2m,0)$, $1 < m < n < 250$ is

$$(0,0) \to (1,0) \to (1,1) \to (2,1) \to (2,2) \to zigzag \to (2m,1) \to (2m{+}1,1) \to (2m{+}2,2)$$

$$\to \cdots \to (499,1) \to (500,1) \to (500,0) \to (499,0) \to \cdots \to (2m+1,0) \to (2m,0).$$

The Hamiltonian paths in all other cases are totally similar to above constructed. We are done.