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Problem Of The Month

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Problem:

99 dwarves have in total 166 hats. Each of these hats belongs to only one of the dwarves and is painted into one of 99 colors. Several festivals with participation of all 99 dwarves are organized. In each festival each dwarf wears one of its hats. No dwarves wear the same coloured hats in any given festival. For any pair of festivals there is at least one dwarf which wears differently coloured hats in these festivals. Find the maximal possible number of organized festivals.

Solution: Answer: 2^{33}. Let us numerate dwarves and colours by numbers $1, 2, \ldots, 99$. Suppose that for $i = 1, 2, \ldots, 33$ each dwarf numbered $2i-1$ and $2i$ have two hats numbered $2i-1$ and $2i$ and for $i = 67, 68, \ldots, 99$ each dwarf numbered i has a hat numbered i. Then by using of these $33 \cdot 2 + 66 = 165$ hats one can organize 2^{33} festivals.

Now let us show that the total number of festivals can not exceed 2^{33}. We prove if n dwarves have in total $n+m$ hats then the total number of festivals can not exceed $2^{\lfloor m/2 \rfloor}$. The statement will be proved by induction over n. If $n = 1$ there is only one colour and one can organize only one festival. Since $1 \leq 2^{\lfloor m/2 \rfloor}$ the statement is true. Assume the inductive hypothesis is held for $1, \ldots, n-1$. Suppose that the dwarf having least number of hats has a hats. For each colour r let $f(r)$ be the total number of hats coloured r. Let $b = min_r f(r)$, where the minimum is taken over all colours r.

Case 1: $a \leq b$. Consider a festival in which the dwarf A having a hats wears a hat coloured s. Then the remaining $n-1$ dwarves can wear at most $n+m-(a+b-1) \leq n+m-(2a-1)$ hats. Since $(n+m-(2a-1))-(n-1) = m-2(a-1)$, by the induction hypothesis the total number of festivals can not exceed $2^{\lfloor m/2 \rfloor - (a-1)}$. Since A has a different options the total number of festivals can not exceed $a2^{\lfloor m/2 \rfloor - (a-1)}$. Now since $a2^{1-a} \leq 1$ we get the desired boundary $2^{\lfloor m/2 \rfloor}$.

Case 2: $a > b$. Consider a festival in which some dwarf A wears a hat coloured r. Then the remaining $n-1$ dwarves can wear at most $n+m-(b+a-1) \leq n+m-(2b-1)$ hats. Since $n+m-(2b-1)-(n-1) = m-2(b-1)$, by the induction hypothesis the total number of festivals can not exceed $2^{\lfloor m/2 \rfloor-(b-1)}$. Since only b dwarves can wear hats coloured r the total number of festivals can not exceed $b2^{\lfloor m/2 \rfloor-(b-1)}$. Now since $b2^{1-b} \leq 1$ we get the desired boundary $2^{\lfloor m/2 \rfloor}$.

When $n = 99$ and $n+m = 166$ we get the boundary $2^{\lfloor 67/2 \rfloor} = 2^{33}$.