Bilkent University
Department of Mathematics

Problem Of The Month

February 2020

Problem:

Each unit square of 2020×2020 grid is painted into either red or white colour. Suppose that among four unit squares belonging to the intersection of any two lines and any two columns there are two same-coloured unit squares lying in the same line or in the same column. Find the minimal possible value of the total number of mono-coloured lines and columns.

Solution: Answer: 2.

Suppose that each unit square of 2020×2020 grid is painted into either red or white colour. If for any two lines and any two columns among four unit squares belonging to the intersection of them there are two same-coloured unit squares lying in the same line or in the same column we say that the colouring is *good*.

Let us show that any good coloured grid contains at least two mono-coloured lines. Suppose that the total number of mono-coloured lines and columns of 2020×2020 grid is 1. Without loss of generality suppose that it is a white coloured line L. A unit square lying in the intersection of line A and column B we denote by (A, B). Let M be a line containing maximal number of red coloured unit squares. Since M is not mono-coloured there is a column and S such that (M, S) is white coloured. Since S is not mono-coloured there exists a line $N \neq L$ such that (N, S) is red coloured. Then since M contains maximal number of red coloured unit squares there exists a column T such that (M, T) is red and (N, T) is white. Therefore, lines M, N and columns S, T do not satisfy the conditions and the colouring is not good. Similarly we can prove that the colouring without mono-coloured lines and columns is not good.

Let A be a fixed line and B be a fixed column of 2020×2020 grid. Suppose that all unit squares of A and B are red coloured and the remaining unit squares are coloured white. The the grid colouring is good and the total number of mono-coloured lines is 2. Done.