Bilkent University
Department of Mathematics

Problem Of The Month

February 2020

Problem:

There are 55! empty boxes numbered 1 to 55! In each step we choose the empty box with minimal number, put one ball into it and after that from each box with number less than the number of chosen box (if any) we transfer one ball to the chosen one. Determine the number of the non-empty box with minimal number and the amount of balls in it after 55! steps.

Note: The number of balls in boxes numbered 1,2,3,4,5 after k steps for $k = 1, 2, ..., 7$ is given below:

1,0,0,0,...
0,2,0,0,...
1,2,0,0,...
0,1,3,0,0,...
1,1,3,0,0,...
0,0,2,4,0,...
1,0,2,4,0,...

Solution: Answer: After 55! steps the number of the non-empty box with minimal number is 58 and the box contains 10 balls.

Let m be any positive integer. Let A_m be the number of balls in the box numbered m and T_m be the total number of balls in the boxes numbered at least m. Then readily, $A_m \leq m$ and T_m is a multiple of m. Indeed, each time when we fill some box numbered $n \geq m$, T_m increases by m.

Let k be the number of the non-empty box with minimal number after 55! steps. Let us show that $k = 58$. Since after each even numbered step the box numbered 1 is empty we get $k > 1$. Suppose that for some $l < 58$ all balls numbered by $1, 2, ..., l-1$ are empty. Let us show that the box numbered l is also empty. Indeed, $A_l + T_{l+1} = 55!$ Therefore, since

$l+1$ divides T_{l+1} and $55!$ we get that $l+1$ divides $A_l < l+1$. Thus, $A_l = 0$. Similarly, $A_{58} \neq 0$, since otherwise $T_{59} = 58!$ and 59 divides $58!$ Now since $A_{58} + T_{59} = 55!$ we get that $A_{58} = 55! \pmod{59}$. By Wilson's theorem $-1 = 55!\cdot 56\cdot 57\cdot 58 = 55!\cdot(-3)\cdot(-2)\cdot(-1) \pmod{59}$. Therefore, $k = A_{58} = 10$.