Bilkent University
Department of Mathematics

Problem Of The Month

January 2020

Problem:

Let $m = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ be the prime decomposition of a positive integer m and the "derivative" function $f(n)$ be defined by

$$f(m) = f(p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}) = d_1 d_2 \cdots d_k p_1^{d_1-1} p_2^{d_2-1} \cdots p_k^{d_k-1}.$$

For a given positive integer L, the L "derivative" sequence is the sequence $\{a_n\}, n = 1, 2, \ldots$ defined by $a_1 = L$ and $a_{n+1} = f(a_n)$, $n > 1$.

We say that a sequence $\{a_n\}$ is not N repeating if $i \neq j$, $a_i = a_j$ implies that $min(i, j) > N$.

Prove or disprove that for each positive N there is a L "derivative" sequence which is not N repeating.

Solution: Answer: For each positive N there is a L "derivative" sequence which is not N repeating.

Let us define a sequence $\{b_n\}$ by $b_1 = 1$, $b_{k+1} = f(b_k)(N - k + 1)$ for $k = 2, ..., N - 1$ and $b_{k+1} = f(b_k)$ for $k \geq N$.

If $\{b_n\}$ is not periodic, then for $b_i \neq b_j$ for all $i, j \geq N$ and for $L = f(b_N)$ the sequence $\{a_n\}$ is L "derivative" sequence which is not N repeating.

If $\{b_n\}$ is periodic then it contains finite number of distinct terms. Therefore, there is a prime number p such that no term of $\{b_n\}$ is divisible by p. Define a "derivative" sequence $\{a_n\}$ by $L = p^N$. Then $a_n = b_n p^{N-n+1}$ for $n = 1, ..., N + 1$ and $a_n = b_n$ for $n \geq N + 2$. For $1 \leq i < j \leq N + 1$ prime decompositions of a_i and a_j contain different number of p factors and for $i \geq N + 2$, a_i is not divisible by p. Therefore, $\{a_n\}$ is a L "derivative" sequence which is not N repeating.