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Problem Of The Month

May 2019

Problem:

Let $S = \{1, 2, \ldots, 2019\}$ and $A_1, A_2, \ldots, A_n$ be subsets of S such that the union of any three of them is equal to S and the union of any two of them is not equal to S. Find the maximal possible value of n.

Solution: Answer: 64.

Assume that $n \geq 65$. Then there are at least $\binom{65}{2} = 2080$ unordered pairs of subsets A_i, A_j. By conditions to each unordered pair A_i, A_j we can correspond an element $c(i,j) \in S$ such that $c(i,j) \notin A_i \cup A_j$. Since $2080 > 2019$ for some A_k, A_l and A_p, A_q we have $c(k,l) = c(p,q)$. Finally, since at least three of the indices k, l, p, q are distinct, the union of some three subsets is not equal to S, a contradiction.

Now let us give an example for $n = 64$. There are $\binom{64}{2} = 2016$ unordered pairs of distinct indices i, j, $1 \leq i, j \leq 64$. Let us fix any one-to-one correspondence between the set of 2016 unordered pairs and the set $\{1, 2, \ldots, 2016\}$: $A_i, A_j \leftrightarrow m(i,j)$. Starting with $A_1 = A_2 = \cdots = A_{64} = S$ for each $m = m(i,j) \in \{1, 2, \ldots, 2016\}$ we remove the number m from both A_i and A_j. Since each number $m \in S$ is removed exactly from two subsets, the collection $A_1, A_2, \ldots, A_{64}$ will satisfy the conditions. Done.