Bilkent University
Department of Mathematics

Problem Of The Month

April 2019

Problem:

There are 2019 bags: each bag contains 2019 beads with total weight 2019 gr. In each bag the beads are numbered by $1, 2, \ldots, 2019$. A proper collection is a collection of differently numbered beads containing at most one bead from each bag and having total weight not less than 2019 gr. Find the maximal possible value of k, if one can always choose at least k different proper collections.

Solution: Answer: 2018!.

A *good* collection is a collection of differently numbered beads containing exactly one bead from each bag. Let $(\sigma_1, \sigma_2, \ldots, \sigma_{2019})$ be a good collection, where for each $i = 1, 2, \ldots, 2019$, σ_i is a number of the bead taken from $i-th$ bag. We say that two good collections $(\sigma'_1, \sigma'_2, \ldots, \sigma'_{2019})$ and $(\sigma''_1, \sigma''_2, \ldots, \sigma''_{2019})$ are equivalent if for some integer k and for each $i = 1, 2, \ldots, 2019$ we have $\sigma'_i + k = \sigma''_i \pmod{2019}$. Clearly, each good collection is equivalent to exactly 2018 other good collections and the set of all 2019! good collections can be partitioned into $\dfrac{2019!}{2019} = 2018!$ equivalence classes. The total weight of all 2019 collections belonging to any fixed equivalence class is equal to $2019 \cdot 2019$, since the sum representing the total weight contains the weight of each bead exactly once. Therefore, in each equivalence class there is at least one good collection with total weight not less than 2019. Thus, in each class there is at least one proper collection and there are at least 2018! distinct proper collections.

Now we give an example when the total number of proper collections is exactly 2018!. Suppose that the weights of the beads of first bag are

$$1 + \frac{2018}{2019}, 1 - \frac{1}{2019}, 1 - \frac{1}{2019}, \cdots, 1 - \frac{1}{2019},$$

respectively and the weights of all other beads are equal to 1. Now note that

(†) any proper collection should contain the first bead of the first bag, since otherwise its total weight is at most $2019 - \frac{1}{2019} < 2019$.

(† †) any proper collection should contain 2019 beads, since otherwise its total weight is at most

$$1 + \frac{2018}{2019} + 2017 = 2019 - \frac{1}{2019} < 2019.$$

Therefore, any proper collection contains the first bead of the first bag and 2018 additional beads and the total number of proper collections is 2018!