Bilkent University
Department of Mathematics

Problem Of The Month

May 2018

Problem:

$n \geq 3$ boxes are placed around a circle. At the first step we choose some boxes. At the second step for each chosen box we put a ball into the chosen box and into each of the two neighboring boxes. Find the total number of possible distinct ball distributions which can be obtained in this way. (All balls are identical.)

Solution: The answer is $2^n - 3 \cdot 2^{n/3} + 2$ for $n = 3k$ and 2^n for $n \neq 3k$.

The number of possible box choices is 2^n. Let us examine cases when two or more different choices produce the same ball distribution. If different choices C_1 and C_2 produce the same distribution then these choices should not coincide on any two consecutive boxes. Otherwise C_1 and C_2 will coincide for all remaining boxes. Suppose that the boxes are numbered in clockwise order as $1, 2, \ldots n$ and the sum $i + j$ is defined in $\pmod{n}$. If in choice C box m is chosen we write $C(m) = X$, otherwise we write $C(m) = Y$. Thus, for two consecutive boxes $m, m+1$ there are four possibilities XX, XY, YX, YY. Note that if $C_1(m) = X, C_1(m+1) = X$ and $C_2(m) = Y, C_2(m+1) = Y$ then C_1 and C_2 can not produce the same distribution, since box m will contain different number of balls. Therefore, there are at most three pairwise different choices producing the same distribution. It can be readily shown that if n is a multiple of 3 then there are only two triples producing the same distribution:

$$\begin{aligned} C_1 &= ...XXYXXYXXYXXY... \\ C_2 &= ...XYXXYXXYXXYX... \\ C_3 &= ...YXXYXXYXXYXX... \end{aligned} \tag{1}$$

$$\begin{aligned} C_1 &= ...YYXYYXYYXYYX... \\ C_2 &= ...YXYYXYYXYYXY... \\ C_3 &= ...XYYXYYXYYXYY... \end{aligned} \tag{2}$$

In each of this two cases each choice can be obtained by shifting of any other one. Evidently if n is not a multiple of 3 there is no any such triple.

Now let us examine the cases when exactly two choices produce the same distribution. Note that for C_1 and C_2 there are two consecutive boxes m and $m+1$ such that $C_1(m) \neq C_2(m)$ and $C_1(m+1) \neq C_2(m+1)$. Otherwise for some m we have $C_1(m) = C_2(m), C_1(m+2) = C_2(m+2)$, $C_1(m+4) = C_2(m+3), \ldots$ and consecutively C_1 will coincide with C_2. W.L.O.G. let $C_1(m) = X, C_1(m+1) = Y$ and $C_2(m) = Y, C_2(m+1) = X$. Then we get $C_1(m+2) = C_2(m+2) = Z$, where Z is either X or Y. Similarly, $C_1(m+3) = X, C_1(m+4) = Y$ and $C_2(m+3) = Y, C_2(m+4) = X$. Then we get $C_1(m+5) = C_2(m+5) = Z$, where Z is either X or Y. Continuing this way we get that starting from box m the choices C_1 and C_2 are $XYZ_1XYZ_2XYZ_3XYZ_4...$ and $YXZ_1YXZ_2YXZ_3YXZ_4...$, respectively. If n is not multiple of 3 then we can not cyclically close this chain and will get a contradiction.

Thus, if n is not a multiple of 3 there are no different choices producing the same distribution and the answer is 2^n.

If n is a multiple of 3 then the positions of Z_i can be determined in 3 ways. If for some choice $Z_1 = Z_2 = Z_3...$ then we get one of the six choices from (1) and (2). Therefore, for each of $3 \cdot (2^{n/3} - 2) \cdot 2$ choices there exists exactly one other choice producing the same distribution. Therefore, the answer is

$$2^n - 6 - 6 \cdot (2^{n/3} - 2) + \frac{6}{3} + \frac{6}{2} \cdot (2^{n/3} - 2) = 2^n - 3 \cdot 2^{n/3} + 2.$$