Bilkent University
Department of Mathematics

Problem Of The Month

April 2018

Problem:

We say that a group of 18 students is a *team* if any two students in this group are friends. It is known that in the school any student belongs to at least one team but if any two students end their friendships at least one student does not belong to any team. We say that a team is *special* if at least one student of the team has no friend outside of this team. Show that any two friends belong to some special team.

Solution:

Let us prove that any two friends A and B belong to some special team. Let us define a longest sequence $S_1, S_2, \ldots, S_m$ of students such that

• for all $1 \leq i < j \leq m$ we have $S_i \neq S_j$

•• for each $1 \leq i \leq m$ any team containing S_i also contains $A, B, S_1, S_2, \ldots, S_{i-1}$.

Note that the sequence is well defined since it contains at least one element. Indeed, if A and B end their friendship then some team S_1 does not belong to any team. Therefore any team containing S_1 contains both A and B (S_1 may coincide with A or B).

Let T be any team of S_m. Assume that S_m has a friend S' outside of T. If S_m and S' end their friendship then there is a student S'' which does not belong to any team. Therefore, any team containing S'' contains also S_m and consequently contains $A, B, S_1, S_2, \ldots, S_{m-1}$. Note that S'' does not coincide with $S_1, S_2, \ldots, S_{m-1}$ since any team of S_m contains $S_1, S_2, \ldots, S_{m-1}$ and any team of S'' contains S'. Then S'' can be added to the sequence $S_1, S_2, \ldots, S_m$. This contradicts the maximality of this sequence. Thus, the team T is special (and it is the only team of S_m). Done.