Bilkent University
Department of Mathematics

Problem Of The Month

June 2017

Problem:

In a country consisting of 2017 cities there are two way flights between some pairs of cities so that any two cities are connected by a flight route (direct or consisting of several flights). Find the minimal possible value of k for which for any such flight arrangement one can declare k cities *special* so that from any city (special or not special) there is a direct flight to at least one special city.

Solution: Answer: $k = 1344$.

Let us reformulate the problem in terms of graph theory. Find the minimal value of k if k vertices of any connected graph on 2017 vertices can be marked so that any vertex has a marked neighbor.

Let us show that $k \geq 1344$. Consider a graph G with vertices v_0, $v_1^j, v_2^j, v_3^j, j = 1, ..., 672$ and edges (v_0, v_1^j), (v_1^j, v_2^j), (v_2^j, v_3^j) $j = 1, ..., 672$. Consider any triple $v_1^j, v_2^j, v_3^j, j = 1, ..., 672$. Then v_2^j should be marked as the only neighbor of v_3^j and one of two neighbors of v_2^j also should be marked. Therefore, at least $2 \cdot 672 = 1344$ vertices should be marked.

Now let us show that $k \leq 1344$. Readily this can be done only for the case when G is a tree. We will color each vertex of G into one of three colors: 0,1,2. Let us color any vertex of G with degree one into color 0. In each further step we color all non-colored neighbors of all colored vertices by coloring the neighbors of vertex colored i into color $i+1$ (mod 3). After at most 2016 steps all vertices will be colored and at least 673 vertices will be colored into one of the colors. We mark all vertices colored into the two other colors. Then there are at most 1344 marked vertices and obviously any vertex with degree at least two will have a marked neighbor. Suppose that a vertex v with degree one is colored i and its only neighbor v' (colored $i-1$) is not marked (w.l.o.g. we suppose that v is not the first colored vertex). Then v is marked and v' will have a marked vertex v'' (colored $i-2$). Now by marking v' and unmarking v the total number of marked vertices will not be changed and the vertex v will have a marked neighbor v'. If necessary, we repeat this process of mark switching for other vertices of degree one and after finite number of steps get a proper marking. Done.