Bilkent University
Department of Mathematics

Problem Of The Month

April 2017

Problem:

For each real number t let $g(t)$ be the total number of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xy + f(y)) = f(x)y + t$$

for all real numbers x, y. Determine the function $g(t)$.

Solution:

The function $f(x) = 0$, for all $x \in \mathbb{R}$ is a solution for $t = 0$. Assume that there exists a number x_0 such that $f(x_0) \neq 0$. Set $x = x_0$ to get $f(x_0 y + f(y)) = f(x_0)y + t$. This shows that f is onto.

Setting $x = 0$ yields $f(f(y)) = f(0)y + t$. If $f(0) = 0$, then $f(f(y)) = t$ and since f is onto, we get $f(x) = t$, for all $x \in \mathbb{R}$ which is a solution iff $t = 0$. Assume $f(0) \neq 0$ which implies that f is one-to-one.

Since f is onto, there exists a number c satisfying $f(c) = 0$. Set $x = c$ and obtain $f(cy + f(y)) = t$. We know that f is one-to-one and hence $cy + f(y)$ should be constant for all $y \in \mathbb{R}$. This implies that $f(y) = k - cy$, for all $y \in \mathbb{R}$ for some constants k, c. By putting this in the original equation we have $c^2 = k$ and $k - kc = t$. Therefore we get $t = c^2 - c^3$. It is clear that $g(c) = c^2 - c^3 = c^2(1 - c)$ is decreasing on $[-\infty, 0]$ and $[1, \infty)$. By AM-GM for $0 < c < 1$, we have $g(c) \leq 4/27$. Thus, for $t < 0$ and $t > 4/27$, there exists a unique c satisfying $t = c^2 - c^3$ and for $0 < t < 4/27$, there exist three such c values. For $t = 0$ and $t = 4/27$, there are exactly two such c values:

$$g(t) = \begin{cases} 1 & \text{if } t < 0, \ \ t > \frac{4}{27} \\ 2 & \text{if } \ t = 0, \ \ t = \frac{4}{27} \\ 3 & \text{if } \ 0 < t < \frac{4}{27} \end{cases}$$